Chapter 1: INTRODUCTION

1.1 Newton's second law can be expressed as

$$\mathbf{F} = m\mathbf{a} \tag{1}$$

where $\mathbf{F}$ is the net force acting on the body, m mass of the body, and $\mathbf{a}$ the acceleration of the body in the direction of the net force. Use Eq. (1) to determine the governing equation of a free-falling body. Consider only the forces due to gravity and the air resistance, which is assumed to be proportional to the square of the velocity of the falling body.

Solution: From the free-body-diagram shown in Fig.P1.1 it follows that

$$m\frac{dv}{dt} = F_g - F_d, \quad F_g = mg, \quad F_d = cv^2$$

where v is the downward velocity (m/s) of the body, F_g is the downward force (N or kg m/s^2) due to gravity, F_d is the upward drag force, m is the mass (kg) of the body, g the acceleration (m/s^2) due to gravity, and c is the proportionality constant (drag coefficient, kg/s). The equation of motion is

$$\frac{dv}{dt} + \alpha v^2 = g, \quad \alpha = \frac{c}{m}$$

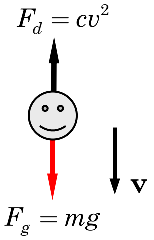

Fig. P1.1

1.2 Consider steady-state heat transfer through a cylindrical bar of nonuniform cross section. The bar is subject to a known temperature T_0 (°C) at the left end and exposed, both on the surface and at the right end, to a medium (such as cooling fluid or air) at temperature T_∞. Assume that temperature is uniform at any section of the bar, $T = T(x)$, and neglect thermal expansion of the bar (that is, assume rigid). Use the principle of conservation of energy (which requires that the rate of change (increase) of internal energy is equal to the sum of heat gained by conduction, convection, and internal heat generation) to a typical element of the bar (see Fig. P1.2) to derive the governing equations of the problem.
Solution: If q denotes the heat flux (heat flow per unit area, W/m^2), then $[Aq]_x$ is the net heat flow into the volume element at x, $[Aq]_{x+\Delta x}$ is the net heat flow out of the volume element at $x + \Delta x$. If h denotes the film conductance [W/(m^2·°C)], $\beta P \Delta x(T_\infty - T)$ is the heat flow through the surface of the rod into the body, where T_∞ is the temperature of the surrounding medium and P is the perimeter (m). Suppose that there is a heat source within the rod generating energy at a rate of g (W/m^3). Then the energy balance gives

$$[Aq]_x - [Aq]_{x+\Delta x} + \beta P\,\Delta x(T_\infty - T) + gA\,\Delta x = 0 \tag{1}$$

or, dividing throughout by Δx,

$$-\frac{[Aq]_{x+\Delta x} - [Aq]_x}{\Delta x} + \beta P(T_\infty - T) + Ag = 0$$

and taking the limit $\Delta x \to 0$, we obtain

$$-\frac{d}{dx}(Aq) + \beta P(T_\infty - T) + Ag = 0 \tag{2}$$

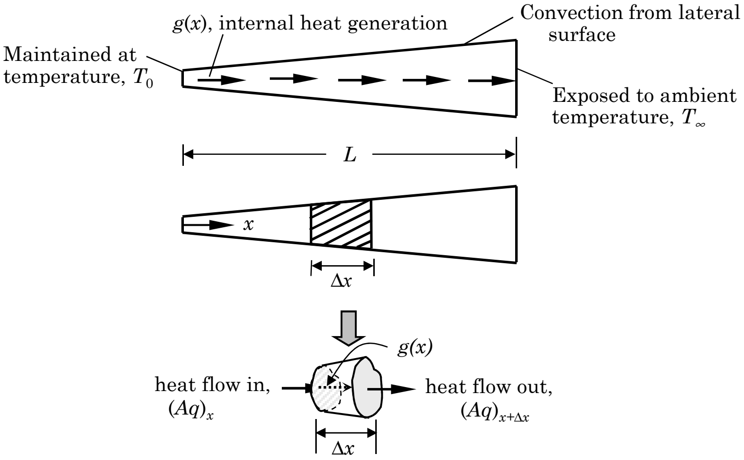

Fig. P1.2

1.3 The Euler–Bernoulli hypothesis concerning the kinematics of bending deformation of a beam assumes that straight lines perpendicular to the beam axis before deformation remain (1) straight, (2) perpendicular to the tangent line to the beam axis, and (3) inextensible during deformation. These assumptions lead to the following displacement field:

$$u_1(x,y) = -y\frac{dv}{dx}, \quad u_2 = v(x), \quad u_3 = 0, \tag{1}$$

where (u_1, u_2, u_3) are the displacements of a point (x, y, z) along the x, y, and z coordinates, respectively, and v is the vertical displacement of the beam at point $(x, 0, 0)$. Suppose that the beam is subjected to a distributed transverse load $q(x)$. Determine the governing equation by summing the forces and moments on an element of the beam (see Fig. P1.3). Note that the sign conventions for the moment and shear force are based on the definitions

$$V = \int_A \sigma_{xy}\, dA, \quad M = \int_A y\, \sigma_{xx}\, dA,$$

and may not agree with the sign conventions used in some mechanics of materials books.

Solution: Summation of the forces in the transverse direction on the element of the beam gives

$$(V + \Delta V) - V + q(x)\Delta x = 0.$$

Dividing throughout with Δx and taking the limit $\Delta x \to 0$ gives

$$\frac{dV}{dx} + q = 0. \tag{2}$$

Taking the moment of forces about the right end of the element, we obtain

$$\sum M_z = 0: \quad -V\Delta x - M + (M + \Delta M) + q\Delta x \cdot \alpha\Delta x = 0,$$

where α is a number $0 \leq \alpha \leq 1$. Again, dividing throughout with Δx and taking the limit $\Delta x \to 0$ gives

$$\frac{dM}{dx} - V = 0. \tag{3}$$

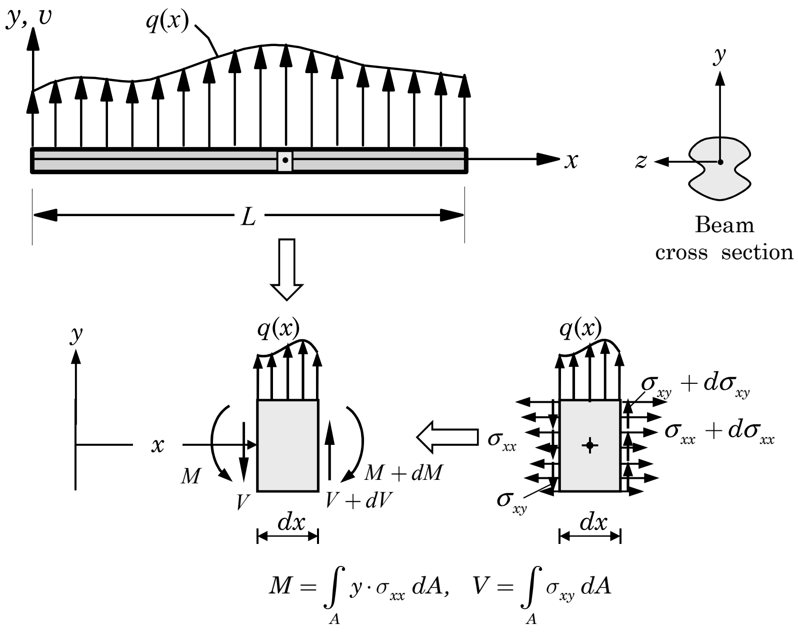

Fig. P1.3

Note that V and M denote the shear force and bending moment on the entire cross section, and they have the meaning

$$M(x) = \int_A \sigma_{xx} y \; dA, \quad V(x) = \frac{dM}{dx}.$$

Here A denotes the area of cross section. The stress resultants (V, M) can be related to the deflection v. Using the linear elastic constitutive relation for an isotropic material

$$\sigma_{xx} = E\varepsilon_{xx} = E\left(-y\frac{d^2v}{dx^2}\right).$$

Substituting into the definition of M, we obtain

$$M(x) = \int_A \sigma_{xx} y \; dA = E\int_A \left(-y\frac{d^2v}{dx^2}\right) y \; dA = -EI\frac{d^2v}{dx^2}, \tag{4}$$

where I is the moment of inertia about the axis of bending ($z-$axis). Then

$$V = -\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right). \tag{5}$$

Equations (2)-(5) can be combined to obtain the following fourth-order equation for v:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = q(x). \tag{6}$$

1.4 A cylindrical storage tank of diameter D contains a liquid column height $h(x, t)$. Liquid is supplied to the tank at a rate of q_i (m^3/day) and drained at a rate of q_0 (m^3/day). Assume that the fluid is incompressible (that is, constant mass density ρ) and use the principle of conservation of mass to obtain a differential equation governing $h(x, t)$.

Solution: The conservation of mass requires

time rate of change in mass = mass inflow - mass outflow.

The above statement for the problem at hand becomes

$$\frac{d}{dt}(\rho A h) = \rho q_i - \rho q_0 \quad \text{or} \quad \frac{d(Ah)}{dt} = q_i - q_0,$$

where A is the area of cross section of the tank ($A = \pi D^2/4$) and ρ is the mass density of the liquid.

1.5 (*Surface tension*). Forces develop at the interface between two immiscible liquids, causing the interface to behave as if it were a membrane stretched over the fluid mass. Molecules in the interior of the fluid mass are surrounded by molecules that are attracted to each other, whereas molecules along the surface (that is, inside the imaginary membrane) are subjected a net force toward the interior. This force imbalance creates a tensile force in the membrane and is called *surface tension* (measured per unit length). Let the difference between the pressure inside the drop and the external pressure be p and the surface tension t_s. Determine the relation between p and t_s for a spherical drop of radius R.

Solution: Consider the free-body-diagram of of a half drop of liquid, as shown in Fig. P1.5. The force due to p is $p(\pi R^2)$, whereas the force in the surface is $t_s(2\pi R)$. The force balance requires

$$p(\pi R^2) = t_s(2\pi R) \quad \Rightarrow \quad p = \frac{2t_s}{R}\,.$$

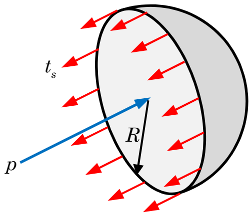

Fig. P1.5

Chapter 2: VECTORS AND TENSORS

2.1 Find the equation of a line (or a set of lines) passing through the terminal point of a vector **A** and in the direction of vector **B**.

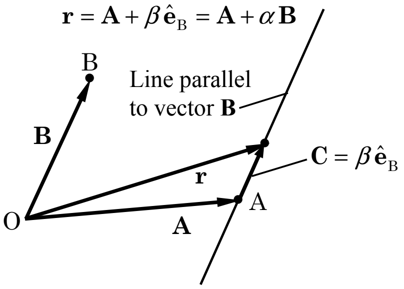

Fig. P2.1

Solution: Let **C** be a vector along the line passing through the terminal point of vector **A** and parallel to vector **B**. Let **r** be the position vector to an arbitrary point on the line parallel to vector **B** and passing through the terminal point of vector **A**. Then the desired equation of the line is (see Fig. P2.1)

$$\mathbf{r} = \mathbf{A} + \mathbf{C} = \mathbf{A} + \beta\,\hat{\mathbf{e}}_B, \quad \hat{\mathbf{e}}_B = \frac{\mathbf{B}}{|\mathbf{B}|},$$

where β is a real number.

2.2 Obtain the equation of a plane perpendicular to a vector **A** and passing through the terminal point of vector **B**, without using any coordinate system.

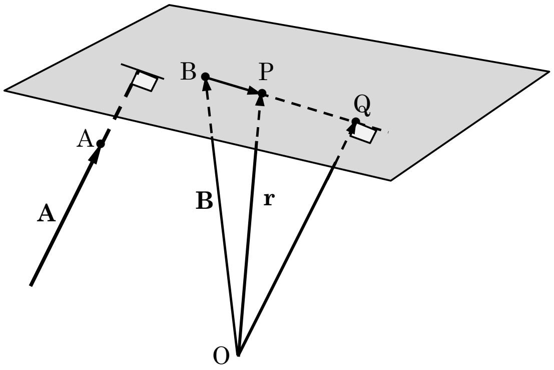

Fig. P2.2

Solution: Let O be the origin and B the terminal point of vector **B**. Draw a directed line segment from O to Q, such that OQ is parallel to vector **A** and point Q is in the plane, as shown in Fig. P2.2. Then OQ is equal to $\alpha\mathbf{A}$, where α is a scalar. Let P be an arbitrary point on the line BQ. If the position vector of the point P is **r**, then the vector connecting points B and P is

$$\mathbf{BP} = \mathbf{r} - \mathbf{B}.$$

Because **BP** is perpendicular to $\mathbf{OQ} = \alpha\mathbf{A}$, we must have

$$\mathbf{BP} \cdot \mathbf{OQ} = 0 \quad \Rightarrow \quad (\mathbf{r} - \mathbf{B}) \cdot \mathbf{A} = 0,$$

which is the required equation of the plane.

2.3 Find the equation of a plane connecting the terminal points of vectors **A**, **B**, and **C**. Assume that all three vectors are referred to a common origin.

Solution: Let **r** denote the position vector. The vectors connecting the terminal points of vectors **A**, **B**, **C**, and **r** should be in the plane. Thus, for example, the scalar triple product of the vectors $\mathbf{B}-\mathbf{A}$, $\mathbf{C}-\mathbf{A}$, and $\mathbf{r}-\mathbf{A}$ should be zero in order that they are co-planar, as shown in Fig. P2.3:

$$(\mathbf{C}-\mathbf{A})\times(\mathbf{B}-\mathbf{A})\cdot(\mathbf{r}-\mathbf{A})=0 \quad [\text{or} \quad e_{ijk}(C_i-A_i)(B_j-A_j)(x_k-A_k)=0]$$

For example, if $\mathbf{A}=\hat{\mathbf{e}}_1$, $\mathbf{B}=\hat{\mathbf{e}}_2$, and $\mathbf{C}=\hat{\mathbf{e}}_3$, then the equation of the plane is $-x-y-z+1=0$ or $x+y+z=1$.

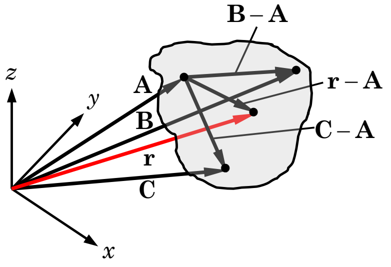

Fig. P2.3

2.4 Let A and B denote two points in space, and let these points be represented by two vectors **A** and **B** with a common origin O, as shown in Fig. P2.4. Show that the straight line through points A and B can be represented by the vector equation

$$(\mathbf{r}-\mathbf{A})\times(\mathbf{B}-\mathbf{A})=\mathbf{0}.$$

Solution: Here we use the fact that when two vectors are parallel their vector product is zero. Because the vectors $\mathbf{r}-\mathbf{A}$ and $\mathbf{B}-\mathbf{A}$ are parallel, their vector product should be zero, giving the required result.

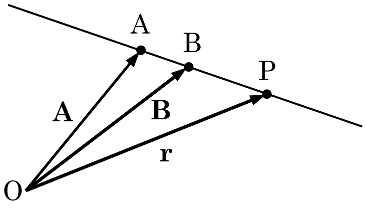

Fig. P2.4

2.5 Prove with the help of vectors that the diagonals of a parallelogram bisect each other.

Solution: Consider the parallelogram formed by points O, A, C, and B, as shown in Fig. P2.5. Let us denote the line segment connecting O to A as vector **A**, O to B as vector **B**, O to C as vector **C**, and B to A as vector **D**. Suppose that vectors **C** and **D** intersect and cross at distances $\alpha\,\mathbf{C}$ and $\beta\,\mathbf{D}$. Then we have the following relations among the four vectors:

$$\mathbf{A}=\alpha\,\mathbf{C}+(1-\beta)\mathbf{D}; \quad \mathbf{A}=\beta\,\mathbf{D}+(1-\alpha)\mathbf{C} \tag{1}$$

$$\mathbf{B} = \alpha\,\mathbf{C} - \beta\,\mathbf{D}; \quad \mathbf{B} = (1-\alpha)\mathbf{C} - (1-\beta)\mathbf{D} \tag{2}$$

From each pair of equations, we obtain the same result, namely, $\alpha = \beta = 0.5$, implying that vectors **C** and **D** bisect each other.

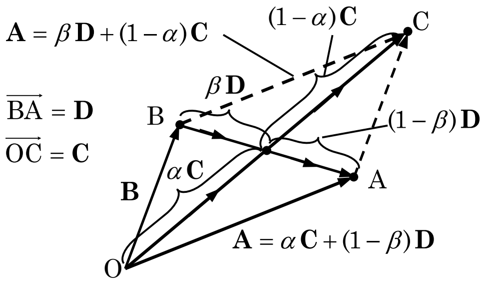

Fig. P2.5

2.6 Show that the position vector **r** that divides a line PQ in the ratio $k : l$ is given by

$$\mathbf{r} = \frac{l}{k+l}\mathbf{A} + \frac{k}{k+l}\mathbf{B},$$

where **A** and **B** are the vectors that designate points P and Q, respectively.

Solution: Let R denote the point on line PQ where the position vector divides it in the ratio $k : l$, as shown in Fig. P2.6, and let $\hat{\mathbf{e}}$ denote the unit vector along the line. Then we have the relations

$$\mathbf{A} + k\,\hat{\mathbf{e}} = \mathbf{r}, \quad \mathbf{B} = \mathbf{r} + l\,\hat{\mathbf{e}}.$$

To eliminate $\hat{\mathbf{e}}$ from the equations, we multiply the first one with l and the second one with k and add the result to obtain

$$l\,\mathbf{A} + kl\,\hat{\mathbf{e}} + k\,\mathbf{B} = (l+k)\mathbf{r} + lk\,\hat{\mathbf{e}} \quad \text{or} \quad \mathbf{r} = \frac{l}{k+l}\mathbf{A} + \frac{k}{k+l}\mathbf{B},$$

which is the desired result.

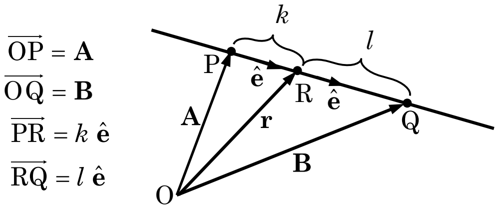

Fig. P2.6

2.7 Represent a tetrahedron by the three non-coplanar vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, as shown in Fig. P2.7. Show that the vectorial sum of the areas of the tetrahedron sides is zero.

Solution: Recall the fact that the cross product $\mathbf{A}\times\mathbf{B}$ vectorially represents the area of the parallelogram formed by the two vectors, which is half the area of the face formed by vectors $\mathbf{A}$ and $\mathbf{B}$ of the tetrahedron. Thus, we can write the vector sum of the areas of the four faces (with normal coming out of each face) as

$$\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A}+(\mathbf{C}-\mathbf{A})\times(\mathbf{B}-\mathbf{A})\,,$$

which is zero because terms cancel out.

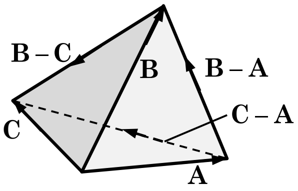

Fig. P2.7

2.8 Deduce that the vector equation for a sphere with its center located at point A and with a radius R is given by

$$(\mathbf{r}-\mathbf{A})\cdot(\mathbf{r}-\mathbf{A})=R^2,$$

where $\mathbf{A}$ is the vector connecting the origin to point A and $\mathbf{r}$ is the position vector.

Solution: The required result follows from the fact that $\mathbf{r}-\mathbf{A}$ is the radius vector, whose magnitude is R, as shown in Fig. P2.8.

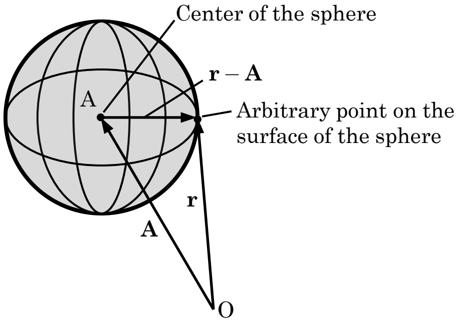

Fig. P2.8

2.9 Verify that the following identity holds (without using index notation):

$$(\mathbf{A}\cdot\mathbf{B})^2+(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{B})=|\mathbf{A}|^2\,|\mathbf{B}|^2,$$

where $\mathbf{A}$ and $\mathbf{B}$ are arbitrary vectors. *Hint:* Use Eqs. (2.2.21) and (2.2.25).

Solution: Let $\mathbf{C}=\mathbf{A}\times\mathbf{B}$, and consider the expression

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}),\tag{1}$$

where we used the identity in Eq. (2.2.21). Then using Eq. (2.2.25), we can write

$$\mathbf{B}\times\mathbf{C}=\mathbf{B}\times(\mathbf{A}\times\mathbf{B})=(\mathbf{B}\cdot\mathbf{B})\mathbf{A}-(\mathbf{B}\cdot\mathbf{A})\mathbf{B}.\tag{2}$$

Hence, we have

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{A}\cdot[(\mathbf{B}\cdot\mathbf{B})\mathbf{A}-(\mathbf{B}\cdot\mathbf{A})\mathbf{B}] = |\mathbf{A}|^2|\mathbf{B}|^2-(\mathbf{A}\cdot\mathbf{B})^2,$$

from which the desired identity follows.

2.10 If **A**, **B**, and **C** are noncoplanar vectors (that is, **A**, **B**, and **C** are linearly independent), determine if the following set of vectors is linearly independent:

$$\mathbf{r}_1 = \mathbf{A}-3\mathbf{B}+2\mathbf{C},\quad \mathbf{r}_2 = 2\mathbf{A}-5\mathbf{B}+3\mathbf{C},\quad \mathbf{r}_3 = \mathbf{A}-5\mathbf{B}+4\mathbf{C}.$$

Solution: Write the linear relation among the three vectors,

$$\alpha\,\mathbf{r}_1+\beta\,\mathbf{r}_2+\gamma\,\mathbf{r}_3 = \mathbf{0},$$

which gives

$$(\alpha+2\beta+\gamma)\mathbf{A}-(3\alpha+5\beta+5\gamma)\mathbf{B}+(2\alpha+3\beta+4\gamma)\mathbf{C} = \mathbf{0}.$$

Because **A**, **B**, and **C** are linearly independent and yet the linear relation must hold implies that the coefficients of the three vectors should be identically zero, giving the following three relations among the real numbers α, β, and γ:

$$\alpha+2\beta+\gamma = 0,\quad 3\alpha+5\beta+5\gamma,\quad 2\alpha+3\beta+4\gamma = 0,$$

whose solution is

$$\alpha = -5\gamma,\quad \beta = 2\gamma,$$

and γ is arbitrary. Hence, the set is linearly dependent. In fact, we can write $-5\mathbf{r}_1+2\mathbf{r}_2+\mathbf{r}_3 = \mathbf{0}$.

2.11 Determine whether the following set of vectors is linearly independent:

$$\mathbf{A} = 2\hat{\mathbf{e}}_1-\hat{\mathbf{e}}_2+\hat{\mathbf{e}}_3,\quad \mathbf{B} = -\hat{\mathbf{e}}_2-\hat{\mathbf{e}}_3,\quad \mathbf{C} = -\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2.$$

Here $\hat{\mathbf{e}}_i$ are orthonormal unit base vectors in $\Re^3$.

Solution: Set the linear combination of the vectors to zero

$$\alpha\mathbf{A}+\beta\mathbf{B}+\gamma\mathbf{C} = \mathbf{0},$$

which gives (if a vector is zero then all its components are zero)

$$2\alpha-\gamma = 0,\quad -\alpha-\beta+\gamma = 0,\quad \alpha-\beta = 0,$$

whose solution is $\gamma = 2\alpha = 2\beta$. Therefore, the linear relation is *not* trivial. The vectors are linearly dependent. In fact, we can write vector **C** as

$$\mathbf{C} = -\frac{1}{2}\left(\mathbf{A}+\mathbf{B}\right).$$

Note that the vectors

$$\mathbf{A} = 2\hat{\mathbf{e}}_1-\hat{\mathbf{e}}_2+\hat{\mathbf{e}}_3,\quad \hat{\mathbf{B}} = \hat{\mathbf{e}}_2-\hat{\mathbf{e}}_3,\quad \mathbf{C} = -\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2$$

are linearly independent.

2.12 Let the vectors $(\hat{\mathbf{i}},\hat{\mathbf{j}},\hat{\mathbf{k}})$ constitute an orthonormal basis. In terms of this basis, define a cogredient basis by

$$\mathbf{e}_1 = -\hat{\mathbf{i}}-\hat{\mathbf{j}},\quad \mathbf{e}_2 = \hat{\mathbf{i}}+2\hat{\mathbf{j}}-2\hat{\mathbf{k}},\quad \mathbf{e}_3 = 2\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}.$$

Determine

(a) the dual or reciprocal (contragredient) basis $(\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3)$ in terms of the orthonormal basis $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$,

(b) the magnitudes (or norms) $|\mathbf{e}_1|$, $|\mathbf{e}_2|$, $|\mathbf{e}_3|$, $|\mathbf{e}^1|$, $|\mathbf{e}^2|$, and $|\mathbf{e}^3|$, and

(c) the cogredient components A_1, A_2, and A_3 of a vector $\mathbf{A}$ if its contragredient components are given by $A^1 = 1, A^2 = 2, A^3 = 3$.

Solution: (a) Note that

$$\mathbf{e}_1 \times \mathbf{e}_2 = 2\,\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} - \hat{\mathbf{k}}, \quad \mathbf{e}_2 \times \mathbf{e}_3 = 4\,\hat{\mathbf{i}} - 5\,\hat{\mathbf{j}} - 3\hat{\mathbf{k}}, \quad \mathbf{e}_3 \times \mathbf{e}_1 = \hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}, \quad [\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3] = 1.$$

Then

$$\mathbf{e}^1 = 4\,\hat{\mathbf{i}} - 5\,\hat{\mathbf{j}} - 3\hat{\mathbf{k}}, \quad \mathbf{e}^2 = \hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}, \quad \mathbf{e}^3 = 2\,\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} - \hat{\mathbf{k}}.$$

(b) The magnitudes of the base vectors are

$$|\mathbf{e}_1| = \sqrt{2}\,, \quad |\mathbf{e}_2| = 3\,, \quad |\mathbf{e}_3| = \sqrt{6}, \quad |\mathbf{e}^1| = 5\sqrt{2}\,, \quad |\mathbf{e}^2| = \sqrt{3}\,, \quad |\mathbf{e}^3| = 3.$$

(c) We have $\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3 = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$. We obtain

$$\begin{aligned} A_1 &= (\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3) \cdot \mathbf{e}_1 = 2 - 2 \times 3 - 3 \times 3 = -13 \\ A_2 &= (\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3) \cdot \mathbf{e}_2 = -3 + 2 \times 9 + 3 \times 2 = 21 \\ A_3 &= (\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3) \cdot \mathbf{e}_3 = -3 + 2 \times 2 + 3 \times 6 = 19. \end{aligned}$$

2.13 Using the Gram–Schmidt orthonormalization process, construct the orthonormal sets associated with the following sets of vectors:

(a) $\mathbf{e}_1 = \hat{\mathbf{i}}_1 + \hat{\mathbf{i}}_3, \qquad \mathbf{e}_2 = \hat{\mathbf{i}}_1 + 2\hat{\mathbf{i}}_2 + 2\hat{\mathbf{i}}_3, \qquad \mathbf{e}_3 = 2\hat{\mathbf{i}}_1 - \hat{\mathbf{i}}_2 + \hat{\mathbf{i}}_3.$

(b) $\mathbf{e}_1 = 2\hat{\mathbf{i}}_1 + \hat{\mathbf{i}}_2, \qquad \mathbf{e}_2 = \hat{\mathbf{i}}_1 - 2\hat{\mathbf{i}}_2 + \hat{\mathbf{i}}_3, \qquad \mathbf{e}_3 = -2\hat{\mathbf{i}}_1 + \hat{\mathbf{i}}_2 + \hat{\mathbf{i}}_3.$

where $(\hat{\mathbf{i}}_1,\, \hat{\mathbf{i}}_2,\, \hat{\mathbf{i}}_3)$ is a an orthonormal Cartesian basis.

Solution: (a) Set

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}_1}{|\mathbf{e}_1|} = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{i}}_1 + \hat{\mathbf{i}}_3\right)$$

Then $\hat{\mathbf{e}}_2$ is constructed as follows:

$$\mathbf{e}_2' = \mathbf{e}_2 - (\hat{\mathbf{e}}_1 \cdot \mathbf{e}_2)\hat{\mathbf{e}}_1 = \hat{\mathbf{i}}_1 + 2\hat{\mathbf{i}}_2 + 2\hat{\mathbf{i}}_3 - \frac{3}{2}\left(\hat{\mathbf{i}}_1 + \hat{\mathbf{i}}_3\right) = \frac{1}{2}\left(-\hat{\mathbf{i}}_1 + 4\hat{\mathbf{i}}_2 + \hat{\mathbf{i}}_3\right).$$

$$\hat{\mathbf{e}}_2 = \frac{\mathbf{e}_2'}{|\mathbf{e}_2'|} = \frac{1}{3\sqrt{2}}\left(-\hat{\mathbf{i}}_1 + 4\hat{\mathbf{i}}_2 + \hat{\mathbf{i}}_3\right).$$

Similarly,

$$\begin{aligned} \mathbf{e}_3' &= \mathbf{e}_3 - (\hat{\mathbf{e}}_1 \cdot \mathbf{e}_3)\hat{\mathbf{e}}_1 - (\hat{\mathbf{e}}_2 \cdot \mathbf{e}_3)\hat{\mathbf{e}}_2 \\ &= 2\hat{\mathbf{i}}_1 - \hat{\mathbf{i}}_2 + \hat{\mathbf{i}}_3 - \frac{3}{2}\left(\hat{\mathbf{i}}_1 + \hat{\mathbf{i}}_3\right) + \frac{5}{18}\left(-\hat{\mathbf{i}}_1 + 4\hat{\mathbf{i}}_2 + \hat{\mathbf{i}}_3\right) \\ &= \frac{1}{9}\left(2\hat{\mathbf{i}}_1 + \hat{\mathbf{i}}_2 - 2\hat{\mathbf{i}}_3\right). \end{aligned}$$

and

$$\hat{\mathbf{e}}_3 = \frac{\mathbf{e}_3'}{|\mathbf{e}_3'|} = \frac{1}{3}\left(2\hat{\mathbf{i}}_1 + \hat{\mathbf{i}}_2 - 2\hat{\mathbf{i}}_3\right).$$

(b) Set

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}_1}{|\mathbf{e}_1|} = \frac{1}{\sqrt{5}}\left(2\hat{\mathbf{i}}_1 + \hat{\mathbf{i}}_2\right)$$

Then $\hat{\mathbf{e}}_2$ is constructed as follows:

$$\mathbf{e}_2' = \mathbf{e}_2 - (\hat{\mathbf{e}}_1 \cdot \mathbf{e}_2)\hat{\mathbf{e}}_1 = \hat{\mathbf{i}}_1 - 2\hat{\mathbf{i}}_2 + \hat{\mathbf{i}}_3 - 0 \quad \Rightarrow \quad \hat{\mathbf{e}}_2 = \frac{1}{\sqrt{6}}\left(\hat{\mathbf{i}}_1 - 2\hat{\mathbf{i}}_2 + \hat{\mathbf{i}}_3\right).$$

Finally,

$$\begin{aligned}\mathbf{e}_3' &= \mathbf{e}_3 - (\hat{\mathbf{e}}_1 \cdot \mathbf{e}_3)\hat{\mathbf{e}}_1 - (\hat{\mathbf{e}}_2 \cdot \mathbf{e}_3)\hat{\mathbf{e}}_2 \\ &= -2\hat{\mathbf{i}}_1 + \hat{\mathbf{i}}_2 + \hat{\mathbf{i}}_3 + \frac{3}{5}\left(2\hat{\mathbf{i}}_1 + \hat{\mathbf{i}}_2\right) + \frac{1}{2}\left(\hat{\mathbf{i}}_1 - 2\hat{\mathbf{i}}_2 + \hat{\mathbf{i}}_3\right) \\ &= \frac{3}{10}\left(-\hat{\mathbf{i}}_1 + 2\hat{\mathbf{i}}_2 + 5\hat{\mathbf{i}}_3\right),\end{aligned}$$

and

$$\hat{\mathbf{e}}_3 = \frac{\mathbf{e}_3'}{|\mathbf{e}_3'|} = \frac{1}{\sqrt{30}}\left(-\hat{\mathbf{i}}_1 + 2\hat{\mathbf{i}}_2 + 5\hat{\mathbf{i}}_3\right).$$

2.14 Prove the following vector identity using index notation:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}.$$

Solution: We begin with the left side of the equality and arrive at the right side:

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= A_i\hat{\mathbf{e}}_i \times (\hat{\mathbf{e}}_\ell e_{jk\ell} B_j C_k) = A_i B_j C_k e_{jk\ell} e_{pi\ell} \hat{\mathbf{e}}_p \\ &= (\delta_{jp}\delta_{ki} - \delta_{ji}\delta_{kp}) A_i B_j C_k \hat{\mathbf{e}}_p = A_i B_j C_i \hat{\mathbf{e}}_j - A_i B_i C_k \hat{\mathbf{e}}_k \\ &= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}.\end{aligned}$$

2.15 Prove the following vector identity using index notation:

$$(\mathbf{A} \cdot \mathbf{B})^2 + (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) = |\mathbf{A}|^2\, |\mathbf{B}|^2.$$

Solution: We begin with

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) &= (A_i B_j e_{ijk} \hat{\mathbf{e}}_k) \cdot (A_m B_n e_{mnp} \hat{\mathbf{e}}_p) \\ &= A_i B_j A_m B_n e_{ijk} e_{mnk} = A_i B_j A_m B_n (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) \\ &= A_i A_i\, B_j B_j - A_i B_i\, A_j B_j = (\mathbf{A} \cdot \mathbf{A})(\mathbf{B} \cdot \mathbf{B}) - (\mathbf{A} \cdot \mathbf{B})(\mathbf{A} \cdot \mathbf{B}),\end{aligned}$$

from which the desired identity follows.

2.16 Use index notation and the e-δ identity to rewrite the vector expression as a sum (or difference) of two vector expressions:

$$(\boldsymbol{\nabla} \times \mathbf{A}) \times \mathbf{B}$$

where **A** and **B** are vector functions.

Solution: We have

$$\begin{aligned}(\boldsymbol{\nabla} \times \mathbf{A}) \times \mathbf{B} &= \varepsilon_{ijk} \frac{\partial A_j}{\partial x_i} \hat{\mathbf{e}}_k \times (B_p \hat{\mathbf{e}}_p) \\ &= \varepsilon_{ijk}\varepsilon_{kpq} \frac{\partial A_j}{\partial x_i} B_p \hat{\mathbf{e}}_q = (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) \frac{\partial A_j}{\partial x_i} B_p \hat{\mathbf{e}}_q \\ &= \frac{\partial A_j}{\partial x_i} B_i \hat{\mathbf{e}}_j - \frac{\partial A_j}{\partial x_i} B_j \hat{\mathbf{e}}_i = \mathbf{B} \cdot \boldsymbol{\nabla}\mathbf{A} - \boldsymbol{\nabla}\mathbf{A} \cdot \mathbf{B}.\end{aligned}$$

2.17 Simplify the vector expression $\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{x}-\mathbf{y}}{\rho}\right)$, where $\rho = |\mathbf{x} - \mathbf{y}|$ and $\mathbf{y}$ is a fixed point, and $\mathbf{x}$ is the position vector of a point in a 3D space. Express the final result in terms of ρ only.

Solution: We have

$$\begin{aligned}\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{x}-\mathbf{y}}{\rho}\right) &= \hat{\mathbf{e}}_j \frac{\partial}{\partial x_j} \cdot \left\{(\mathbf{x} - \mathbf{y})[(x_i - y_i)(x_i - y_i)]^{-1/2}\right\} \\ &= \hat{\mathbf{e}}_j \cdot \left\{\mathbf{e}_j [(x_i - y_i)(x_i - y_i)]^{-1/2}\right\}\end{aligned}$$

$$
\begin{aligned}
&+\hat{\mathbf{e}}_j \cdot \left\{ (\mathbf{x}-\mathbf{y}) \Big(-\frac{1}{2}\Big) [(x_i-y_i)(x_i-y_i)]^{-3/2} \frac{\partial}{\partial x_j} [(x_i-y_i)(x_i-y_i)] \right\} \\
&= \frac{3}{\rho} - \hat{\mathbf{e}}_j \cdot \left\{ \frac{\mathbf{x}-\mathbf{y}}{\rho^3}(x_j-y_j) \right\} = \frac{3}{\rho} - \left\{ \frac{(x_j-y_j)}{\rho^3}(x_j-y_j) \right\} = \frac{2}{\rho}
\end{aligned}
$$

2.18 Using index notation prove the following identities among vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$:

$$
\begin{aligned}
&\text{(a)} \quad (\mathbf{A}\times\mathbf{B})\cdot(\mathbf{B}\times\mathbf{C})\times(\mathbf{C}\times\mathbf{A}) = (\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}))^2. \\
&\text{(b)} \quad (\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = [\mathbf{A}\cdot(\mathbf{C}\times\mathbf{D})]\mathbf{B} - [\mathbf{B}\cdot(\mathbf{C}\times\mathbf{D}]\mathbf{A}.
\end{aligned}
$$

Solution: (**a**) We have $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{B}\times\mathbf{C})\times(\mathbf{C}\times\mathbf{A})$

$$
\begin{aligned}
&= (e_{ijk}A_iB_j\hat{\mathbf{e}}_k)\cdot[(e_{rst}B_rC_s\hat{\mathbf{e}}_t)\times(e_{mnp}C_mA_n\hat{\mathbf{e}}_p)] \\
&= e_{ijk}e_{rst}e_{mnp}A_iB_jB_rC_sC_mA_ne_{tpq}\delta_{kq} \\
&= (\delta_{it}\delta_{jp}-\delta_{ip}\delta_{jt})\,e_{rst}e_{mnp}A_iB_jB_rC_sC_mA_n \\
&= (e_{rsi}e_{mnj}-e_{rsj}e_{mni})\,A_iB_jB_rC_sC_mA_n \\
&= (\mathbf{B}\times\mathbf{C}\cdot\mathbf{A})\,(\mathbf{C}\times\mathbf{A}\cdot\mathbf{B}) - (\mathbf{B}\times\mathbf{C}\cdot\mathbf{B})\,(\mathbf{C}\times\mathbf{A}\cdot\mathbf{A}) \\
&= (\mathbf{B}\times\mathbf{C}\cdot\mathbf{A})\,(\mathbf{C}\times\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\times\mathbf{C}\cdot\mathbf{A})^2
\end{aligned}
$$

Note that $\mathbf{B}\times\mathbf{C}\cdot\mathbf{A} = \mathbf{A}\times\mathbf{B}\cdot\mathbf{C}$.

(**b**) We have

$$
\begin{aligned}
(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) &= (e_{ijk}A_iB_j\hat{\mathbf{e}}_k)\times(e_{mnp}C_mD_n\hat{\mathbf{e}}_p) \\
&= e_{ijk}e_{kpq}e_{mnp}A_iB_jC_mD_n\hat{\mathbf{e}}_q \\
&= e_{mnp}\,(\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp})\,A_iB_jC_mD_n\hat{\mathbf{e}}_q \\
&= e_{mni}A_iB_jC_mD_n\hat{\mathbf{e}}_j - e_{mnj}A_iB_jC_mD_n\hat{\mathbf{e}}_i \\
&= (\mathbf{C}\times\mathbf{D}\cdot\mathbf{A})\,\mathbf{B} - (\mathbf{C}\times\mathbf{D}\cdot\mathbf{B})\,\mathbf{A}.
\end{aligned}
$$

2.19 Prove that

$$
[\mathbf{ABC}][\mathbf{DEF}] = \begin{vmatrix} \mathbf{A}\cdot\mathbf{D} & \mathbf{A}\cdot\mathbf{E} & \mathbf{A}\cdot\mathbf{F} \\ \mathbf{B}\cdot\mathbf{D} & \mathbf{B}\cdot\mathbf{E} & \mathbf{B}\cdot\mathbf{F} \\ \mathbf{C}\cdot\mathbf{D} & \mathbf{C}\cdot\mathbf{E} & \mathbf{C}\cdot\mathbf{F} \end{vmatrix},
$$

and from there show that

$$
e_{ijk}\,e_{rst} = \begin{vmatrix} \delta_{ir} & \delta_{is} & \delta_{it} \\ \delta_{jr} & \delta_{js} & \delta_{jt} \\ \delta_{kr} & \delta_{ks} & \delta_{kt} \end{vmatrix}.
$$

Solution: Recall the two properties of determinants: (1) det $([S][T])$ = det $[S]\cdot$ det $[T]$ and (2) det $[S]^{\mathrm{T}}$= det $[S]$. Therefore, we have

$$
\begin{aligned}
[\mathbf{ABC}][\mathbf{DEF}] &= \left| \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \begin{bmatrix} D_1 & E_1 & F_1 \\ D_2 & E_2 & F_2 \\ D_3 & E_3 & F_3 \end{bmatrix} \right| \\
&= \begin{vmatrix} A_1D_1+A_2D_2+A_3D_3 & A_1E_1+A_2E_2+A_3E_3 & A_1F_1+A_2F_2+A_3F_3 \\ B_1D_1+B_2D_2+B_3D_3 & B_1E_1+B_2E_2+B_3E_3 & B_1F_1+B_2F_2+B_3F_3 \\ C_1D_1+C_2D_2+C_3D_3 & C_1E_1+C_2E_2+C_3E_3 & C_1F_1+C_2F_2+C_3F_3 \end{vmatrix} \\
&= \begin{vmatrix} \mathbf{A}\cdot\mathbf{D} & \mathbf{A}\cdot\mathbf{E} & \mathbf{A}\cdot\mathbf{F} \\ \mathbf{B}\cdot\mathbf{D} & \mathbf{B}\cdot\mathbf{E} & \mathbf{B}\cdot\mathbf{F} \\ \mathbf{C}\cdot\mathbf{D} & \mathbf{C}\cdot\mathbf{E} & \mathbf{C}\cdot\mathbf{F} \end{vmatrix}. \qquad (1)
\end{aligned}
$$

The second identity follows from Eq. (1) when we take $\mathbf{A}=\mathbf{D}=\hat{\mathbf{e}}_1$, $\mathbf{B}=\mathbf{E}=\hat{\mathbf{e}}_2$, and $\mathbf{C}=\mathbf{F}=\hat{\mathbf{e}}_3$.

2.20 Establish the following identities :

$$\text{(a)}\quad e_{ijk} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix}. \qquad \text{(b)}\quad e_{ijk}e_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix}.$$

$$\text{(c)}\quad e_{ijk}\, e_{ijk} = 6. \qquad \text{(d)}\quad e_{ijk}e_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}.$$

Solution:

(**a**) Let $\hat{\mathbf{e}}_i = \delta_{ip}\hat{\mathbf{e}}_p$, $\hat{\mathbf{e}}_j = \delta_{jq}\hat{\mathbf{e}}_q$, and $\hat{\mathbf{e}}_k = \delta_{kr}\hat{\mathbf{e}}_r$. Then the determinant form of the triple scalar product $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j \times \hat{\mathbf{e}}_k$ is

$$e_{ijk} = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j \times \hat{\mathbf{e}}_k = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix}. \tag{1}$$

(**b**) This was already established in the previous problem; we show it independently of Problem 2.19. Again, recall the two properties of determinants: (1) det $([S][T])$ = det $[S]\cdot$ det $[T]$ and (2) det $[S]^{\mathrm{T}}$= det $[S]$. Therefore, we begin with Eq. (1):

$$\begin{aligned} e_{ijk}e_{pqr} &= \left| \begin{bmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{bmatrix} \begin{bmatrix} \delta_{p1} & \delta_{q1} & \delta_{r1} \\ \delta_{p2} & \delta_{q2} & \delta_{r2} \\ \delta_{p3} & \delta_{q3} & \delta_{r3} \end{bmatrix} \right| \\ &= \begin{vmatrix} \delta_{i1}\delta_{p1} + \delta_{i2}\delta_{p2} + \delta_{i3}\delta_{p3} & \delta_{i1}\delta_{q1} + \delta_{i2}\delta_{q2} + \delta_{i3}\delta_{q3} & \delta_{i1}\delta_{r1} + \delta_{i2}\delta_{r2} + \delta_{i3}\delta_{r3} \\ \delta_{j1}\delta_{p1} + \delta_{j2}\delta_{p2} + \delta_{j3}\delta_{p3} & \delta_{j1}\delta_{q1} + \delta_{j2}\delta_{q2} + \delta_{j3}\delta_{q3} & \delta_{j1}\delta_{r1} + \delta_{j2}\delta_{r2} + \delta_{j3}\delta_{r3} \\ \delta_{k1}\delta_{p1} + \delta_{k2}\delta_{p2} + \delta_{k3}\delta_{p3} & \delta_{k1}\delta_{q1} + \delta_{k2}\delta_{q2} + \delta_{k3}\delta_{q3} & \delta_{k1}\delta_{r1} + \delta_{k2}\delta_{r2} + \delta_{k3}\delta_{r3} \end{vmatrix} \\ &= \begin{vmatrix} \delta_{im}\delta_{mp} & \delta_{im}\delta_{mq} & \delta_{im}\delta_{mr} \\ \delta_{jm}\delta_{mp} & \delta_{jm}\delta_{mq} & \delta_{jm}\delta_{mr} \\ \delta_{km}\delta_{mp} & \delta_{km}\delta_{mq} & \delta_{km}\delta_{mr} \end{vmatrix} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix}, \end{aligned} \tag{2}$$

where we have used the identity of the form

$$\delta_{i1}\delta_{p1} + \delta_{i2}\delta_{p2} + \delta_{i3}\delta_{p3} = \delta_{im}\delta_{mp} = \delta_{ip}, \quad \text{etc.}$$

(**c**) Using the e-δ identity we can write

$$e_{ijk}\, e_{ijk} = \delta_{jj}\delta_{kk} - \delta_{jk}\delta_{jk} = 9 - 3 = 6.$$

(**d**) From Part (b), we have

$$\begin{aligned} e_{ijk}e_{mnk} &= \begin{vmatrix} \delta_{im} & \delta_{in} & \delta_{ik} \\ \delta_{jm} & \delta_{jn} & \delta_{jk} \\ \delta_{km} & \delta_{kn} & \delta_{kk} \end{vmatrix} \\ &= \delta_{im}\left(\delta_{jn}\delta_{kk} - \delta_{kn}\delta_{jk}\right) - \delta_{jm}\left(\delta_{in}\delta_{kk} - \delta_{kn}\delta_{ik}\right) + \delta_{km}\left(\delta_{in}\delta_{jk} - \delta_{jn}\delta_{ik}\right) \\ &= 3\delta_{im}\delta_{jn} - \delta_{im}\delta_{jn} - 3\delta_{jm}\delta_{in} + \delta_{jm}\delta_{in} + \delta_{in}\delta_{jm} - \delta_{jn}\delta_{im} \\ &= \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}. \end{aligned}$$

2.21 Consider two rectangular Cartesian coordinate systems that are translated and rotated with respect to each other. The transformation between the two coordinate systems is given by

$$\bar{\mathbf{x}} = \mathbf{c} + \mathbf{L}\mathbf{x},$$

where $\mathbf{c}$ is a constant vector and $\mathbf{L} = [\ell_{ij}]$ is the matrix of direction cosines

$$\ell_{ij} \equiv \hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j.$$

Deduce that the following orthogonality conditions hold:

$$\mathbf{L} \cdot \mathbf{L}^{\mathrm{T}} = \mathbf{I}.$$

That is, $\mathbf{L}$ is an orthogonal matrix.

Solution: We have

$$\hat{\hat{\mathbf{e}}}_i = \ell_{ij}\,\hat{\mathbf{e}}_j, \quad \hat{\mathbf{e}}_n = \ell_{mn}\,\hat{\hat{\mathbf{e}}}_m.$$

Then

$$\hat{\mathbf{e}}_i = \ell_{ji}\,\hat{\hat{\mathbf{e}}}_j = \ell_{ji}\,\ell_{jp}\,\hat{\mathbf{e}}_p.$$

Taking dot product with $\hat{\mathbf{e}}_k$ on both sides, we obtain

$$\delta_{ik} = \ell_{ji}\,\ell_{jp}\delta_{pk} = \ell_{ji}\,\ell_{jk} \;\text{ or }\; \delta_{ij} = \ell_{ki}\,\ell_{kj} \;\text{ (renamed } j \text{ as } k \text{ and } k \text{ as } j) \;\text{ or }\; \mathbf{L}\cdot\mathbf{L}^{\mathrm{T}} = \mathbf{I}.$$

2.22 Determine the transformation matrix relating the orthonormal basis vectors $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ and $(\hat{\mathbf{e}}'_1, \hat{\mathbf{e}}'_2, \hat{\mathbf{e}}'_3)$, when $\hat{\mathbf{e}}'_i$ are given by

(a) $\hat{\mathbf{e}}'_1$ is along the vector $\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$ and $\hat{\mathbf{e}}'_2$ is perpendicular to the plane $2x_1 + 3x_2 + x_3 - 5 = 0$.

(b) $\hat{\mathbf{e}}'_1$ is along the line segment connecting point $(1, -1, 3)$ to $(2, -2, 4)$ and $\hat{\mathbf{e}}'_3 = (-\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3)/\sqrt{6}$.

Solution: (**a**) Let $\hat{\mathbf{e}}_i$ be the unit base vectors in the current orthogonal system, and $\hat{\mathbf{e}}'_i$ be the unit base vectors in the new coordinate system. The vector $\hat{\mathbf{e}}'_1$ has the same direction as the vector $\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$ but its magnitude must be unity

$$\hat{\mathbf{e}}'_1 = \frac{\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3}{|\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3|} = \frac{1}{\sqrt{3}}(\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3).$$

The vector $\hat{\mathbf{e}}'_2$ is along the normal to the plane $2x_1 + 3x_2 + x_3 - 5 = 0$. Hence, $\hat{\mathbf{e}}'_2 = \hat{\mathbf{n}}$, the unit normal to the plane, which is given by

$$\begin{aligned}\hat{\mathbf{e}}'_2 &= \frac{\boldsymbol{\nabla}(2x_1 + 3x_2 + x_3 - 5)}{|\boldsymbol{\nabla}(2x_1 + 3x_2 + x_3 - 5)|} = \frac{2\hat{\mathbf{e}}_1 + 3\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3}{\sqrt{(2)^2 + (3)^2 + (1)^2}} \\ &= \frac{1}{\sqrt{14}}(2\hat{\mathbf{e}}_1 + 3\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3).\end{aligned}$$

The third basis vector in an orthonormal system is related to the other two vectors by

$$\begin{aligned}\hat{\mathbf{e}}'_3 = \hat{\mathbf{e}}'_1 \times \hat{\mathbf{e}}'_2 &= \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \end{vmatrix} \\ &= \hat{\mathbf{e}}_1\left(-\frac{1}{\sqrt{42}} - \frac{3}{\sqrt{42}}\right) - \hat{\mathbf{e}}_2\left(\frac{1}{\sqrt{42}} - \frac{2}{\sqrt{42}}\right) + \hat{\mathbf{e}}_3\left(\frac{3}{\sqrt{42}} + \frac{2}{\sqrt{42}}\right) \\ &= \frac{1}{\sqrt{42}}(-4\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + 5\hat{\mathbf{e}}_3).\end{aligned}$$

Thus, the two coordinate systems are related by (note the matrix of direction cosines)

$$\left\{\begin{matrix} \hat{\mathbf{e}}'_1 \\ \hat{\mathbf{e}}'_2 \\ \hat{\mathbf{e}}'_3 \end{matrix}\right\} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \\ \frac{-4}{\sqrt{42}} & \frac{1}{\sqrt{42}} & \frac{5}{\sqrt{42}} \end{bmatrix} \left\{\begin{matrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{matrix}\right\}.$$

(**b**) We have

$$\begin{aligned}
\hat{\mathbf{e}}_1' &= \frac{\text{vector connecting point } (1,-1,3) \text{ to point}(2,-2,4)}{\text{vector magnitude}} \\
&= \frac{(2\hat{\mathbf{e}}_1 - 2\hat{\mathbf{e}}_2 + 4\hat{\mathbf{e}}_3) - (\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + 3\hat{\mathbf{e}}_3)}{\text{magnitude}} = \frac{1}{\sqrt{3}}(\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3) \\
\hat{\mathbf{e}}_3' &= \frac{1}{\sqrt{6}}(-\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3) \quad \text{(given)} \\
\hat{\mathbf{e}}_2' &= \hat{\mathbf{e}}_3' \times \hat{\mathbf{e}}_1' = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{vmatrix} \\
&= \hat{\mathbf{e}}_1\left(\frac{1}{\sqrt{18}} + \frac{2}{\sqrt{18}}\right) - \hat{\mathbf{e}}_2\left(\frac{-1}{\sqrt{18}} - \frac{2}{\sqrt{18}}\right) + \hat{\mathbf{e}}_3\left(\frac{1}{\sqrt{18}} - \frac{1}{\sqrt{18}}\right) \\
&= \frac{1}{\sqrt{2}}\left(\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_3\right).
\end{aligned}$$

Transformation matrix relating $(\hat{\mathbf{e}}_1', \hat{\mathbf{e}}_2', \hat{\mathbf{e}}_3')$ to $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ is given by

$$[L] = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \quad (a_{ij} = \hat{\mathbf{e}}_i' \cdot \hat{\mathbf{e}}_j)$$

2.23 The angles between the barred and unbarred coordinate lines are given by

	$\hat{\mathbf{e}}_1$	$\hat{\mathbf{e}}_2$	$\hat{\mathbf{e}}_3$
$\bar{\hat{\mathbf{e}}}_1$	60°	30°	90°
$\bar{\hat{\mathbf{e}}}_2$	150°	60°	90°
$\bar{\hat{\mathbf{e}}}_3$	90°	90°	0°

Determine the direction cosines of the transformation.

Solution: Follows from the definition

$$[L] = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (a_{ij} = \bar{\hat{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j).$$

2.24 The angles between the barred and unbarred coordinate lines are given by

	x_1	x_2	x_3
$\bar{x}_1$	45°	90°	45°
$\bar{x}_2$	60°	45°	120°
$\bar{x}_3$	120°	45°	60°

Determine the transformation matrix.

Solution: Follows from the definition

$$[L] = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} \quad (\ell_{ij} = \bar{\hat{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j).$$

2.25 Write the following sets of equations in matrix form $[A]\{X\} = \{Y\}$:

$$\text{(a)}\quad \begin{aligned} 2x_1 + x_2 - 2x_3 &= 1, \\ x_1 - 2x_2 + x_3 &= 5, \\ 3x_1 + x_2 - x_3 &= 4. \end{aligned} \qquad \text{(b)}\quad \begin{aligned} 2x_1 + x_2 - x_3 &= 0, \\ 3x_1 \qquad - x_3 &= 2, \\ x_1 + x_2 + x_3 &= 1. \end{aligned}$$

Solution: The matrix representation of the linear equations is

$$\text{(a)}\quad \begin{bmatrix} 2 & 1 & -2 \\ 1 & -2 & 1 \\ 3 & 1 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 5 \\ 4 \end{Bmatrix}, \quad \text{(b)}\quad \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2 \\ 1 \end{Bmatrix}.$$

2.26 Determine the cofactors and the determinants of the coefficient matrices in Problem **2.25**.

Solution: (a) First we compute the adjoints, as shown below.

$$A_{11} = \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} = 1, \quad A_{12} = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = -4, \quad A_{13} = \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} = 7,$$

$$A_{21} = \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} = 1, \quad A_{22} = \begin{vmatrix} 2 & -2 \\ 3 & -1 \end{vmatrix} = 4, \quad A_{23} = \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = -1,$$

$$A_{31} = \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = -3, \quad A_{32} = \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} = 4, \quad A_{33} = \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -5.$$

The determinant is given by (expansion by first row)

$$\begin{aligned} |A| &= (-1)^{1+1} a_{11} A_{11} + (-1)^{1+2} a_{12} A_{12} + (-1)^{1+3} a_{13} A_{13} \\ &= 2 \times 1 - 1 \times (-4) + (-2) \times 7 = -8. \end{aligned}$$

Using the first column, we obtain the same result

$$\begin{aligned} |A| &= (-1)^{1+1} a_{11} A_{11} + (-1)^{1+2} a_{21} A_{21} + (-1)^{1+3} a_{31} A_{31} \\ &= 2 \times 1 - 1 \times 1 + 3 \times (-3) = -8. \end{aligned}$$

(b) The adjoints are

$$A_{11} = \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} = 1, \quad A_{12} = \begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} = 4, \quad A_{13} = \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} = 3,$$

$$A_{21} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2, \quad A_{22} = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3, \quad A_{23} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1,$$

$$A_{31} = \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} = -1, \quad A_{32} = \begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix} = 1, \quad A_{33} = \begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} = -3.$$

The determinant is given by (using the first row)

$$\begin{aligned} |A| &= (-1)^{1+1} a_{11} A_{11} + (-1)^{1+2} a_{12} A_{12} + (-1)^{1+3} a_{13} A_{13} \\ &= 2 \times 1 - 1 \times 4 + (-1) \times 3 = -5. \end{aligned}$$

Using the second column, we obtain the same result

$$\begin{aligned} |A| &= (-1)^{1+2} a_{12} A_{12} + (-1)^{2+2} a_{22} A_{22} + (-1)^{1+3} a_{32} A_{32} \\ &= -1 \times 4 - 1 \times 1 = -5. \end{aligned}$$

2.27 Find the inverses of the coefficient matrices in Problem **2.25**.

Solution: (a) The inverse is given by (A_{ij} are the adjoints defined in the solution to Problem **2.26**)

$$[A]^{-1} = \frac{1}{|A|} \begin{bmatrix} A_{11} & -A_{21} & A_{31} \\ -A_{12} & A_{22} & -A_{32} \\ A_{13} & -A_{23} & A_{33} \end{bmatrix} = -\frac{1}{8} \begin{bmatrix} 1 & -1 & -3 \\ 4 & 4 & -4 \\ 7 & 1 & -5 \end{bmatrix}$$

(b) The inverse is given by

$$[A]^{-1} = \frac{1}{|A|} \begin{bmatrix} A_{11} & -A_{21} & A_{31} \\ -A_{12} & A_{22} & -A_{32} \\ A_{13} & -A_{23} & A_{33} \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} 1 & -2 & -1 \\ -4 & 3 & -1 \\ 3 & -1 & -3 \end{bmatrix}$$

2.28 Determine if the following matrices are positive:

$$(a)\quad \begin{bmatrix} 2 & 1 & -2 \\ 1 & -2 & 1 \\ 3 & 1 & -1 \end{bmatrix}, \quad (b)\quad \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \quad (c)\quad \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 2 & 1 \end{bmatrix}.$$

Solution: (a) We write the quadratic form of the matrix

$$\{X\}^{\mathrm{T}}[A]\{X\} = 2x_1^2 - 2x_2^2 - x_3^2 + 2\left(x_1x_2 + x_1x_3 + x_2x_3\right)$$

which is not positive because the vector, for example, $(x_1, x_2, x_3) = (0, 1, 1)$ gives $\{X\}^{\mathrm{T}}[A]\{X\} = -1$.
(b) The quadratic form

$$\{X\}^{\mathrm{T}}[A]\{X\} = 2(x_1 + x_2)^2 > 0$$

is positive, and hence $[A]$ is positive.
(c) The quadratic form

$$\{X\}^{\mathrm{T}}[A]\{X\} = 2x_1^2 - x_2^2 + x_3^2 + 2x_1x_2 + 4x_1x_3 + 4x_2x_3$$

is not always positive, because $(x_1, x_2, x_3) = (-1, 1, -1)$ gives $\{X\}^{\mathrm{T}}[A]\{X\} = 0$, and $(x_1, x_2, x_3) = (0, 1, -1)$ as well as $(x_1, x_2, x_3) = (0, -1, 1)$ give $\{X\}^{\mathrm{T}}[A]\{X\} = -4$.

2.29 Check to see if the following $[Q]$ is nonsingular, and if it is, construct the positive matrix associated with it:

$$[Q] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Solution: We have

$$|Q| = 1 \times (1 \times 1 - 1 \times 2) + 1 \times (0 \times 2 - 1 \times 0) = -1 \neq 0.$$

Therefore, the positive matrix associated with $[Q]$ is

$$[A] = [Q]^{\mathrm{T}}[Q] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}.$$

2.30 Let $\mathbf{r}$ denote a position vector $\mathbf{r} = \mathbf{x} = x_i\hat{\mathbf{e}}_i$ $(r^2 = x_ix_i)$ and $\mathbf{A}$ an arbitrary constant vector. Show that:

(a) $\nabla^2(r^n) = n(n+1)r^{n-2}$. (b) $\text{grad}\ (\mathbf{r} \cdot \mathbf{A}) = \mathbf{A}$.

(c) $\text{div}\ (\mathbf{r} \times \mathbf{A}) = 0$. (d) $\text{curl}(\mathbf{r} \times \mathbf{A}) = -2\mathbf{A}$.

(e) $\text{div}\ (r\mathbf{A}) = \dfrac{1}{r}(\mathbf{r} \cdot \mathbf{A})$. (f) $\text{curl}\ (r\mathbf{A}) = \dfrac{1}{r}(\mathbf{r} \times \mathbf{A})$.

Solution: First we establish following two identities:

$$\begin{aligned} \text{grad}(r) &= \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}\left(x_jx_j\right)^{\frac{1}{2}} = \hat{\mathbf{e}}_i \frac{1}{2}\left(x_jx_j\right)^{\frac{1}{2}-1} 2x_i \\ &= \hat{\mathbf{e}}_i x_i \left(x_jx_j\right)^{-\frac{1}{2}} = \frac{\mathbf{r}}{r}. \end{aligned} \tag{1}$$

$$\begin{aligned} \text{grad}(r^n) &= \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}\left(x_jx_j\right)^{\frac{n}{2}} = \hat{\mathbf{e}}_i \frac{n}{2}\left(x_jx_j\right)^{\frac{n}{2}-1} 2x_i \\ &= n\hat{\mathbf{e}}_i x_i \left(x_jx_j\right)^{\frac{n-2}{2}} = nr^{n-2}\mathbf{r}. \end{aligned} \tag{2}$$

(**a**) We have,

$$\begin{aligned}\nabla^2(r^n) &= \frac{\partial^2}{\partial x_i \partial x_i}(r^n) = \frac{\partial}{\partial x_i}\left(nr^{n-2}x_i\right)\\ &= n(n-2)r^{n-3}\frac{\partial r}{\partial x_i}x_i + nr^{n-2}\delta_{ii} = n(n-2)r^{n-3}\frac{x_i}{r}x_i + 3nr^{n-2}\\ &= [n(n-2)+3n]r^{n-2} = n(n+1)r^{n-2}.\end{aligned}$$

(**b**) Because **A** is a constant vector, we have

$$\begin{aligned}\text{grad}(\mathbf{r}\cdot\mathbf{A}) &= \hat{\mathbf{e}}_i\frac{\partial}{\partial x_i}(x_jA_j) = \hat{\mathbf{e}}_i\left(\delta_{ij}A_j + x_j\frac{\partial A_j}{\partial x_i}\right)\\ &= \hat{\mathbf{e}}_i(A_i+0) = \mathbf{A}.\end{aligned}$$

(**c**) Carrying out the indicated operation, we obtain

$$\begin{aligned}\text{div}(\mathbf{r}\times\mathbf{A}) &= \hat{\mathbf{e}}_i\cdot\frac{\partial}{\partial x_i}(e_{jk\ell}x_jA_k\hat{\mathbf{e}}_\ell) = e_{jk\ell}\delta_{i\ell}\left(\frac{\partial x_j}{\partial x_i}A_k + x_j\frac{\partial A_k}{\partial x_i}\right)\\ &= (0+0) = 0.\end{aligned}$$

(**d**) Carrying out the indicated operation, we obtain

$$\begin{aligned}\text{curl}(\mathbf{r}\times\mathbf{A}) &= \hat{\mathbf{e}}_i\times\frac{\partial}{\partial x_i}(e_{rst}x_rA_s\hat{\mathbf{e}}_t)\\ &= e_{rst}\hat{\mathbf{e}}_i\times\hat{\mathbf{e}}_t\left(\delta_{ir}A_s + x_r\frac{\partial A_s}{\partial x_i}\right)\\ &= e_{rst}e_{jit}\hat{\mathbf{e}}_j(\delta_{ir}A_s+0) = e_{ist}e_{jit}\hat{\mathbf{e}}_jA_s\\ &= -2\hat{\mathbf{e}}_j\delta_{sj}A_s = -2\mathbf{A}.\end{aligned}$$

(**e**) Carrying out the indicated operation, we obtain

$$\text{div}(r\mathbf{A}) = \hat{\mathbf{e}}_i\cdot\frac{\partial}{\partial x_i}(rA_j\hat{\mathbf{e}}_j) = \hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j\left(\frac{\partial r}{\partial x_i}A_j\right) = \frac{x_i}{r}A_i = \frac{1}{r}\mathbf{r}\cdot\mathbf{A}.$$

(**f**) Carrying out the indicated operation, we obtain

$$\begin{aligned}\boldsymbol{\nabla}\times(r\mathbf{A}) &= \hat{\mathbf{e}}_i\times\frac{\partial}{\partial x_i}\left(\sqrt{x_jx_j}A_k\hat{\mathbf{e}}_k\right)\\ &= \hat{\mathbf{e}}_i\times\hat{\mathbf{e}}_k\left(\frac{x_i}{r}A_k+0\right) = e_{ikt}\hat{\mathbf{e}}_t\left(\frac{x_i}{r}A_k\right)\\ &= \frac{1}{r}e_{ikt}x_iA_k\hat{\mathbf{e}}_t = \frac{1}{r}(\mathbf{r}\times\mathbf{A}).\end{aligned}$$

2.31 Let **A** and **B** be vector functions of position vector **x** with continuous first and second derivatives, and let F and G be scalar functions of position **x** with continuous first and second derivatives. Show that:

(a) $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = 0$.
(b) $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}F) = 0$.
(c) $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}F\times\boldsymbol{\nabla}G) = 0$.
(d) $\boldsymbol{\nabla}\cdot(F\mathbf{A}) = \mathbf{A}\cdot\boldsymbol{\nabla}F + F\,\boldsymbol{\nabla}\cdot\mathbf{A}$.
(e) $\boldsymbol{\nabla}\times(F\mathbf{A}) = F\,\boldsymbol{\nabla}\times\mathbf{A} - \mathbf{A}\times\boldsymbol{\nabla}F$.
(f) $\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\boldsymbol{\nabla}\mathbf{B} + \mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{A} + \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A})$.
(g) $\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = \boldsymbol{\nabla}\times\mathbf{A}\cdot\mathbf{B} - \boldsymbol{\nabla}\times\mathbf{B}\cdot\mathbf{A}$.

Solution:

(**a**) Using the index notation, we write

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) &= \hat{\mathbf{e}}_i\cdot\frac{\partial}{\partial x_i}\left(e_{jk\ell}\frac{\partial A_k}{\partial x_j}\hat{\mathbf{e}}_\ell\right)\\ &= e_{jk\ell}\delta_{i\ell}\frac{\partial^2 A_k}{\partial x_i\partial x_j} = e_{ijk}\frac{\partial^2 A_k}{\partial x_i\partial x_j} = 0,\end{aligned}$$

because of the symmetry of $A_{k,ij}$ in i and j.

(**b**) We have

$$\text{curl}(\text{grad}F) = \hat{\mathbf{e}}_i\frac{\partial}{\partial x_i}\times\left(\hat{\mathbf{e}}_j\frac{\partial F}{\partial x_j}\right) = e_{ijk}\,\hat{\mathbf{e}}_k\,\frac{\partial^2 F}{\partial x_i\partial x_j} = 0,$$

where the last result is arrived by virtue of the symmetry of $(\partial^2 F/\partial x_i\partial x_j) = (\partial^2 F/\partial x_j\partial x_i)$.

(**c**) Because $\frac{\partial^2 F}{\partial x_i\partial x_j}$ is symmetric in i and j and $\frac{\partial^2 G}{\partial x_i\partial x_k}$ is symmetric in i and k, we obtain

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}F\times\boldsymbol{\nabla}G) &= \left(\hat{\mathbf{e}}_i\frac{\partial}{\partial x_i}\right)\cdot\left(\hat{\mathbf{e}}_j\frac{\partial F}{\partial x_j}\times\hat{\mathbf{e}}_k\frac{\partial G}{\partial x_k}\right)\\ &= e_{jk\ell}(\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_\ell)\left(\frac{\partial^2 F}{\partial x_i\partial x_j}\frac{\partial G}{\partial x_k}+\frac{\partial F}{\partial x_j}\frac{\partial^2 G}{\partial x_i\partial x_k}\right)\\ &= e_{ijk}\left(\frac{\partial^2 F}{\partial x_i\partial x_j}\frac{\partial G}{\partial x_k}+\frac{\partial F}{\partial x_j}\frac{\partial^2 G}{\partial x_i\partial x_k}\right) = 0.\end{aligned}$$

(**d**) Because $F = F(\mathbf{x})$ and $\mathbf{A}{=}\mathbf{A}(\mathbf{x})$, we have

$$\begin{aligned}\boldsymbol{\nabla}\cdot(F\mathbf{A}) &= \left(\hat{\mathbf{e}}_i\frac{\partial}{\partial x_i}\right)\cdot(\hat{\mathbf{e}}_j A_j\ F) = \delta_{ij}\left(\frac{\partial A_j}{\partial x_i}F + A_j\frac{\partial F}{\partial x_i}\right)\\ &= \frac{\partial A_i}{\partial x_i}F + A_i\frac{\partial F}{\partial x_i} = \boldsymbol{\nabla}\cdot\mathbf{A}F + \mathbf{A}\cdot\boldsymbol{\nabla}F.\end{aligned}$$

(**e**) Because $F = F(\mathbf{x})$ and $\mathbf{A}{=}\mathbf{A}(\mathbf{x})$, we have

$$\begin{aligned}\boldsymbol{\nabla}\times(F\mathbf{A}) &= \left(\hat{\mathbf{e}}_i\frac{\partial}{\partial x_i}\right)\times(\hat{\mathbf{e}}_j A_j\ F) = e_{ijk}\hat{\mathbf{e}}_k\left(\frac{\partial A_j}{\partial x_i}F + A_j\frac{\partial F}{\partial x_i}\right)\\ &= e_{ijk}\hat{\mathbf{e}}_k\frac{\partial A_j}{\partial x_i}F - e_{jik}\hat{\mathbf{e}}_k A_j\frac{\partial F}{\partial x_i} = F\boldsymbol{\nabla}\times\mathbf{A} - \mathbf{A}\times\boldsymbol{\nabla}F.\end{aligned}$$

(**f**) Because $\mathbf{A} = \mathbf{A}(\mathbf{x})$ and $\mathbf{B} = \mathbf{B}(\mathbf{x})$, we have

$$\begin{aligned}\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) &= \hat{\mathbf{e}}_i\frac{\partial}{\partial x_i}(A_j B_j)\\ &= \hat{\mathbf{e}}_i\left(\frac{\partial A_j}{\partial x_i}B_j + A_j\frac{\partial B_j}{\partial x_i}\right) = \boldsymbol{\nabla}\mathbf{A}\cdot\mathbf{B} + \boldsymbol{\nabla}\mathbf{B}\cdot\mathbf{A}. \qquad (1)\end{aligned}$$

Next consider

$$\begin{aligned}\mathbf{A}\times\text{curl}\,\mathbf{B} &= (\hat{\mathbf{e}}_i A_i)\times\left(e_{jkp}\hat{\mathbf{e}}_p\frac{\partial B_k}{\partial x_j}\right) = \hat{\mathbf{e}}_q e_{qip}e_{jkp}A_i\frac{\partial B_k}{\partial x_j}\\ &= \hat{\mathbf{e}}_q(\delta_{qj}\delta_{ik} - \delta_{qk}\delta_{ij})A_i\frac{\partial B_k}{\partial x_j} = \hat{\mathbf{e}}_j A_i\frac{\partial B_i}{\partial x_j} - \hat{\mathbf{e}}_k A_i\frac{\partial B_k}{\partial x_i}\\ &= \boldsymbol{\nabla}\mathbf{B}\cdot\mathbf{A} - \mathbf{A}\cdot\boldsymbol{\nabla}\mathbf{B}. \qquad (2)\end{aligned}$$

Therefore,

$$\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}\mathbf{B}\cdot\mathbf{A} - \mathbf{A}\cdot\boldsymbol{\nabla}\mathbf{B} + \boldsymbol{\nabla}\mathbf{A}\cdot\mathbf{B} - \mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{A}$$

$$= \boldsymbol{\nabla}\mathbf{A}\cdot\mathbf{B} + \boldsymbol{\nabla}\mathbf{B}\cdot\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla}\mathbf{B} + \mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{A})\,.$$

From Eqs. (1) and (2) the required vector identity follows.

(**g**) Because $\mathbf{A} = \mathbf{A}(\mathbf{x})$ and $\mathbf{B} = \mathbf{B}(\mathbf{x})$, we have

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) &= \hat{\mathbf{e}}_i\cdot\frac{\partial}{\partial x_i}\left(e_{jk\ell}A_jB_k\hat{\mathbf{e}}_\ell\right) = e_{ijk}\left(\frac{\partial A_j}{\partial x_i}B_k + A_j\frac{\partial B_k}{\partial x_i}\right)\\ &= \mathrm{curl}\mathbf{A}\cdot\mathbf{B} - \mathrm{curl}\mathbf{B}\cdot\mathbf{A}.\end{aligned}$$

2.32 Let $\mathbf{A}$ and $\mathbf{B}$ be vector functions of position vector $\mathbf{x}$ with continuous first and second derivatives, and let F and G be scalar functions of position $\mathbf{x}$ with continuous first and second derivatives. Show that:

(a) $\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{A} - \mathbf{A}\cdot\boldsymbol{\nabla}\mathbf{B} + \mathbf{A}\boldsymbol{\nabla}\cdot\mathbf{B} - \mathbf{B}\boldsymbol{\nabla}\cdot\mathbf{A}$.

(b) $(\boldsymbol{\nabla}\times\mathbf{A})\times\mathbf{A} = \mathbf{A}\cdot\boldsymbol{\nabla}\mathbf{A} - \boldsymbol{\nabla}\mathbf{A}\cdot\mathbf{A}$.

(c) $\nabla^2(FG) = F\,\nabla^2G + 2\boldsymbol{\nabla}F\cdot\boldsymbol{\nabla}G + G\,\nabla^2F$.

(d) $\nabla^2(F\mathbf{x}) = 2\boldsymbol{\nabla}F + \mathbf{x}\nabla^2F$.

(e) $\mathbf{A}\cdot\boldsymbol{\nabla}\mathbf{A} = \boldsymbol{\nabla}\big(\tfrac{1}{2}\mathbf{A}\cdot\mathbf{A}\big) - \mathbf{A}\times\boldsymbol{\nabla}\times\mathbf{A}$.

(f) $\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{x}) = \mathbf{A} + \boldsymbol{\nabla}\mathbf{A}\cdot\mathbf{x}$.

(g) $\nabla^2\left(\mathbf{A}\cdot\mathbf{x}\right) = 2\boldsymbol{\nabla}\cdot\mathbf{A} + \mathbf{x}\cdot\nabla^2\mathbf{A}$.

(**a**) Because $\mathbf{A} = \mathbf{A}(\mathbf{x})$ and $\mathbf{B} = \mathbf{B}(\mathbf{x})$, we have

$$\begin{aligned}\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) &= \hat{\mathbf{e}}_i\frac{\partial}{\partial x_i}\times\left(A_jB_ke_{jkp}\hat{\mathbf{e}}_p\right)\\ &= e_{jkp}e_{qip}\hat{\mathbf{e}}_q\left(\frac{\partial A_j}{\partial x_i}B_k + A_j\frac{\partial B_k}{\partial x_i}\right)\\ &= (\delta_{jq}\delta_{ki} - \delta_{ji}\delta_{kq})\hat{\mathbf{e}}_q\left(\frac{\partial A_j}{\partial x_i}B_k + A_j\frac{\partial B_k}{\partial x_i}\right)\\ &= \hat{\mathbf{e}}_j\left(\frac{\partial A_j}{\partial x_i}B_i + A_j\frac{\partial B_i}{\partial x_i}\right) - \hat{\mathbf{e}}_k\left(\frac{\partial A_i}{\partial x_i}B_k + A_i\frac{\partial B_k}{\partial x_i}\right)\\ &= \mathbf{B}\cdot\nabla\mathbf{A} + \mathbf{A}\nabla\cdot\mathbf{B} - \mathbf{B}\nabla\cdot\mathbf{A} - \mathbf{A}\cdot\nabla\mathbf{B}.\end{aligned}$$

(**b**) Because $\mathbf{A} = \mathbf{A}(\mathbf{x})$, we have

$$\begin{aligned}(\boldsymbol{\nabla}\times\mathbf{A})\times\mathbf{A} &= e_{ijk}\frac{\partial A_j}{\partial x_i}\hat{\mathbf{e}}_k\times(A_p\hat{\mathbf{e}}_p)\\ &= e_{ijk}e_{kpq}\frac{\partial A_j}{\partial x_i}A_p\hat{\mathbf{e}}_q = (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\,\frac{\partial A_j}{\partial x_i}A_p\hat{\mathbf{e}}_q\\ &= \frac{\partial A_j}{\partial x_i}A_i\hat{\mathbf{e}}_j - \frac{\partial A_j}{\partial x_i}A_j\hat{\mathbf{e}}_i = \mathbf{A}\cdot\boldsymbol{\nabla}\mathbf{A} - \boldsymbol{\nabla}\mathbf{A}\cdot\mathbf{A}.\end{aligned}$$

(**c**) Because $\nabla^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$, we have

$$\begin{aligned}\nabla^2(FG) &= \hat{\mathbf{e}}_i\frac{\partial}{\partial x_i}\cdot\left(\hat{\mathbf{e}}_j\frac{\partial F}{\partial x_j}G + F\hat{\mathbf{e}}_j\frac{\partial G}{\partial x_j}\right)\\ &= \hat{\mathbf{e}}_i\cdot\left[\hat{\mathbf{e}}_j\left(\frac{\partial^2F}{\partial x_i\partial x_j}G + \frac{\partial F}{\partial x_j}\frac{\partial G}{\partial x_i}\right) + \hat{\mathbf{e}}_j\left(\frac{\partial F}{\partial x_i}\frac{\partial G}{\partial x_j} + F\frac{\partial^2G}{\partial x_j\partial x_i}\right)\right]\\ &= \hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j\frac{\partial^2F}{\partial x_i\partial x_j}G + 2\hat{\mathbf{e}}_j\frac{\partial F}{\partial x_j}\cdot\hat{\mathbf{e}}_i\frac{\partial G}{\partial x_i} + F\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j\frac{\partial^2G}{\partial x_j\partial x_i}\\ &= \frac{\partial^2F}{\partial x_i\partial x_i}G + 2\boldsymbol{\nabla}F\cdot\boldsymbol{\nabla}G + F\frac{\partial^2G}{\partial x_i\partial x_i}\end{aligned}$$