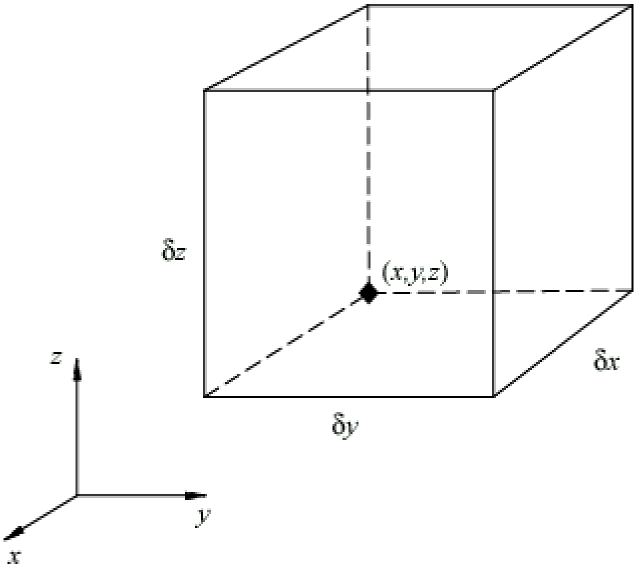
Problem 1.1



Inflow through x = constant: $\rho u\,\delta y\,\delta z$

Outflow through $x+\delta x$ = constant: $\rho u\,\delta y\,\delta z+\dfrac{\partial}{\partial x}(\rho u\,\delta y\,\delta z)\delta x+\cdots$

Net inflow through x = constant surfaces: $-\dfrac{\partial}{\partial x}(\rho u)\delta x\,\delta y\,\delta z+\cdots$

Net inflow through y = constant surfaces: $-\dfrac{\partial}{\partial y}(\rho v)\delta x\,\delta y\,\delta z+\cdots$

Net inflow through z = constant surfaces: $-\dfrac{\partial}{\partial z}(\rho w)\delta x\,\delta y\,\delta z+\cdots$

But the rate at which the mass is accumulating inside the control volume is:

$$\frac{\partial}{\partial t}(\rho\,\delta x\,\delta y\,\delta z)$$

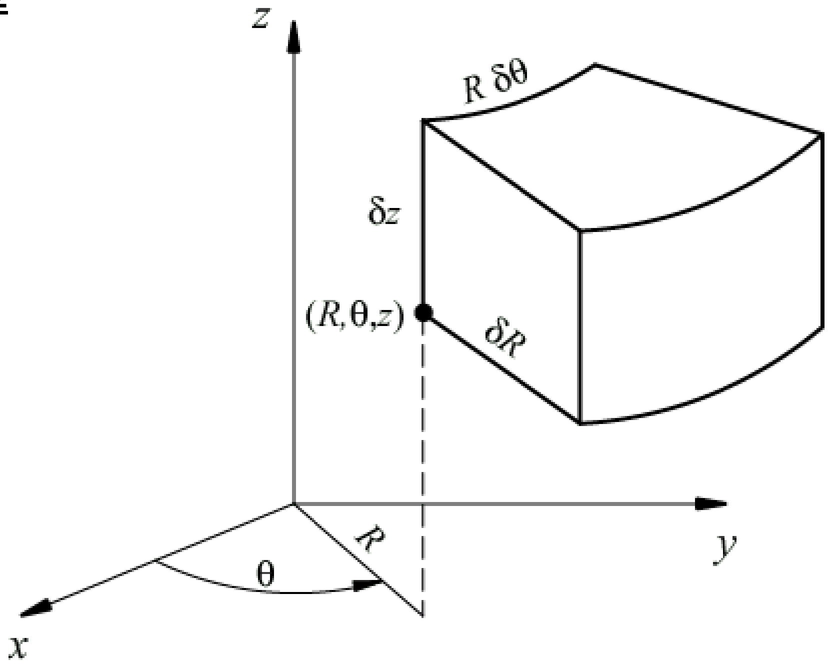
Then the equation of mass conservation becomes:

$$\frac{\partial\rho}{\partial t}\delta x\,\delta y\,\delta z=-\left[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right]\delta x\,\delta y\,\delta z+\cdots$$

Taking the limits as the quantities $\delta x, \delta y$ and δz become vanishingly small, we get:

$$\boxed{\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0}$$

Problem 1.2



Inflow through R = constant: $\rho u_R R\,\delta\theta\,\delta z$

Outflow through $R+\delta R$ = constant: $\rho u_R R\,\delta\theta\,\delta z + \dfrac{\partial}{\partial R}(\rho u_R R\,\delta\theta\,\delta z)\,\delta R + \cdots$

Net inflow through R = constant surfaces: $-\dfrac{\partial}{\partial R}(\rho R u_R)\,\delta R\,\delta\theta\,\delta z + \cdots$

Net inflow through θ = constant surfaces: $-\dfrac{\partial}{\partial \theta}(\rho u_\theta)\,\delta R\,\delta\theta\,\delta z + \cdots$

Net inflow through z = constant surfaces: $-\dfrac{\partial}{\partial z}(\rho u_z)\,R\,\delta R\,\delta\theta\,\delta z + \cdots$

But the rate at which the mass is accumulating inside the control volume is:

$$\frac{\partial}{\partial t}(\rho R\,\delta R\,\delta\theta\,\delta z)$$

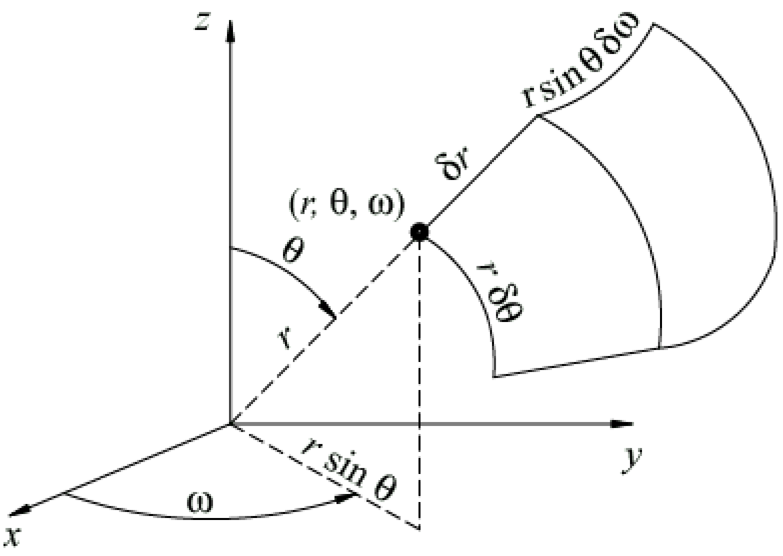
Then the equation of mass conservation becomes:

$$\frac{\partial \rho}{\partial t} R\,\delta R\,\delta\theta\,\delta z = -\left[\frac{\partial}{\partial R}(\rho R u_R) + \frac{\partial}{\partial \theta}(\rho u_\theta) + R\frac{\partial}{\partial z}(\rho u_z)\right]\delta R\,\delta\theta\,\delta z + \cdots$$

Taking the limits as the quantities $\delta R, \delta\theta$ and δz become vanishingly small, we get:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(\rho R u_R) + \frac{1}{R}\frac{\partial}{\partial \theta}(\rho u_\theta) + \frac{\partial}{\partial z}(\rho u_z) = 0}$$

Problem 1.3



Inflow through r = constant: $\rho u_r r^2 \sin\theta\,\delta\theta\,\delta\omega$

Outflow through $r+\delta r$ = constant: $\rho u_r r^2 \sin\theta\,\delta\theta\,\delta\omega$

$$+\frac{\partial}{\partial r}(\rho r^2 u_r \sin\theta\,\delta\theta\,\delta\omega)\delta r + \bullet\bullet\bullet$$

Net inflow through r = constant surfaces: $-\dfrac{\partial}{\partial r}(\rho r^2 u_r)\sin\theta\,\delta r\,\delta\theta\,\delta\omega + \bullet\bullet\bullet$

Net inflow through θ = constant surfaces: $-\dfrac{\partial}{\partial \theta}(\rho u_\theta \sin\theta)\, r\,\delta r\,\delta\theta\,\delta\omega + \bullet\bullet\bullet$

Net inflow through ω = constant surfaces: $-\dfrac{\partial}{\partial \omega}(\rho u_\omega)\, r\,\delta r\,\delta\theta\,\delta\omega + \bullet\bullet\bullet$

But the rate at which the mass is accumulating inside the control volume is:

$$\frac{\partial}{\partial t}(\rho r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\omega)$$

Then the equation of mass conservation becomes:

$$\frac{\partial \rho}{\partial t} r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\omega = -\left[\frac{\partial}{\partial r}(\rho r^2 u_r)\sin\theta + r\frac{\partial}{\partial \theta}(\rho u_\theta \sin\theta) + r\frac{\partial}{\partial \omega}(\rho u_\omega)\right]\delta r\,\delta\theta\,\delta\omega + \bullet\bullet\bullet$$

Taking the limits as the quantities δr, $\delta\theta$ and $\delta\omega$ become vanishingly small, we get:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho u_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \omega}(\rho u_\omega) = 0}$$

Problem 1.4

Using the given transformation equations gives:

$$R^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}$$

$$\therefore 2R\frac{\partial R}{\partial x} = 2x = 2R\cos\theta \qquad \Rightarrow \frac{\partial R}{\partial x} = \cos\theta$$

$$\text{and } \sec^2\theta\frac{\partial\theta}{\partial x} = -\frac{y}{x^2} = -\frac{1}{R}\frac{\sin\theta}{\cos^2\theta} \Rightarrow \frac{\partial\theta}{\partial x} = -\frac{1}{R}\sin\theta$$

Using these results, the derivatives with respect to x and y transform as follows:

$$\frac{\partial}{\partial x} = \frac{\partial R}{\partial x}\frac{\partial}{\partial R} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} = \cos\theta\,\frac{\partial}{\partial R} - \frac{\sin\theta}{R}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial R}{\partial y}\frac{\partial}{\partial R} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} = \sin\theta\,\frac{\partial}{\partial R} + \frac{\cos\theta}{R}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z}$$

Using these results and the relationships between the Cartesian and cylindrical vector components, we get the following expressions for the Cartesian terms in the continuity equation:

$$\frac{\partial}{\partial x}(\rho u) = \cos\theta\frac{\partial}{\partial R}[\rho(u_R\cos\theta - u_\theta\sin\theta)] - \frac{\sin\theta}{R}\frac{\partial}{\partial\theta}[(\rho(u_R\cos\theta - u_\theta\sin\theta)]$$

$$\frac{\partial}{\partial y}(\rho v) = \sin\theta\frac{\partial}{\partial R}[\rho(u_R\sin\theta + u_\theta\cos\theta)] + \frac{\cos\theta}{R}\frac{\partial}{\partial\theta}[(\rho(u_R\sin\theta + u_\theta\cos\theta)]$$

Adding these two terms and simplifying produces the following equation:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = \frac{\partial}{\partial R}(\rho u_R) + \frac{\rho u_R}{R} + \frac{1}{R}\frac{\partial}{\partial\theta}(\rho u_\theta)$$

$$= \frac{1}{R}\frac{\partial}{\partial R}(\rho R u_R) + \frac{1}{R}\frac{\partial}{\partial\theta}(\rho u_\theta)$$

Substituting this result into the full continuity equation yields the following result:

$$\boxed{\frac{\partial\rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(\rho R u_R) + \frac{1}{R}\frac{\partial}{\partial\theta}(\rho u_\theta) + \frac{\partial}{\partial z}(\rho u_z) = 0}$$

Problem 1.5

The equations that connect the two coordinate systems are as follows:

$$x = r\sin\theta\,\cos\omega \qquad y = r\sin\theta\,\sin\omega \qquad z = r\cos\theta$$

$$r^2 = x^2 + y^2 + z^2 \qquad \tan\omega = \frac{y}{x} \qquad \cos^2\theta = \frac{z^2}{(x^2 + y^2 + z^2)}$$

Using the relations given above, the following identities are obtained for the various partial derivatives:

$$\frac{\partial r}{\partial x} = \sin\theta\,\cos\omega \qquad \frac{\partial r}{\partial y} = \sin\theta\,\sin\omega \qquad \frac{\partial r}{\partial z} = \cos\theta$$

$$\frac{\partial \theta}{\partial x} = -\frac{1}{r}\cos\theta\,\cos\omega \qquad \frac{\partial \theta}{\partial y} = -\frac{1}{r}\cos\theta\,\sin\omega \qquad \frac{\partial \theta}{\partial z} = -\frac{1}{r}\sin\theta$$

$$\frac{\partial \omega}{\partial x} = \frac{1}{r}\frac{\sin\omega}{\sin\theta} \qquad \frac{\partial \omega}{\partial y} = \frac{1}{r}\frac{\cos\omega}{\cos\theta} \qquad \frac{\partial \omega}{\partial z} = 0$$

Thus the following expressions are obtained for the various Cartesian derivatives:

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \omega}{\partial x}\frac{\partial}{\partial \omega} \\
&= \sin\theta\,\cos\omega\frac{\partial}{\partial r} - r\cos\theta\,\cos\omega\frac{\partial}{\partial \omega} + \frac{1}{r}\frac{\sin\omega}{\sin\theta}\frac{\partial}{\partial \omega} \\
\frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \omega}{\partial y}\frac{\partial}{\partial \omega} \\
&= \sin\theta\,\sin\omega\frac{\partial}{\partial r} - \frac{1}{r}\cos\theta\,\sin\omega\frac{\partial}{\partial \theta} + \frac{1}{r}\frac{\cos\omega}{\sin\theta}\frac{\partial}{\partial \omega} \\
\frac{\partial}{\partial z} &= \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \omega}{\partial z}\frac{\partial}{\partial \omega} \\
&= \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial \theta}
\end{aligned}$$

Next we need the Cartesian velocity components expressed in terms of spherical components. This may be achieved by noting that the velocity vector may be written as follows:

$$\mathbf{u} = u\,\mathbf{e}_x + v\,\mathbf{e}_y + w\,\mathbf{e}_z = u_r\,\mathbf{e}_r + u_\theta\,\mathbf{e}_\theta + u_\omega\,\mathbf{e}_\omega$$

Then, if we express the base vectors in spherical coordinates in terms of the base vectors in Cartesian coordinates, equating components in the previous equation will yield the required relationships. Thus, noting that:

$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z = r\sin\theta\,\cos\omega\,\mathbf{e}_x + r\sin\theta\,\sin\omega\,\mathbf{e}_y + r\cos\theta\,\mathbf{e}_z$$

Also, recalling from Appendix A that:

$$\mathbf{e}_i = \frac{\partial\mathbf{r}}{\partial x_i}\Bigg/\left|\frac{\partial\mathbf{r}}{\partial x_i}\right| \qquad \text{it follows that}$$

$$\mathbf{e}_r = \sin\theta\,\cos\omega\,\mathbf{e}_x + \sin\theta\,\sin\omega\,\mathbf{e}_y + \cos\theta\,\mathbf{e}_z$$
$$\mathbf{e}_\theta = \cos\theta\,\cos\omega\,\mathbf{e}_x + \cos\theta\,\sin\omega\,\mathbf{e}_y - \sin\theta\,\mathbf{e}_z$$
$$\mathbf{e}_\omega = -\sin\omega\,\mathbf{e}_x + \cos\omega\,\mathbf{e}_y$$

Substituting these expressions into the equation obtained above for the velocity vector, and equating coefficients of like base vectors, yields the following relationships connecting the Cartesian and spherical velocity components:

$$u = u_r\sin\theta\,\cos\omega + u_\theta\cos\theta\,\cos\omega - u_\omega\sin\omega$$
$$v = u_r\sin\theta\,\sin\omega + u_\theta\cos\theta\,\sin\omega + u_\omega\cos\omega$$
$$w = u_r\cos\theta - u_\theta\sin\theta$$

Using these results, and those obtained for the Cartesian derivatives, produces the following expression for the Cartesian terms that appear in the continuity equation:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho u_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\omega}(\rho u_\omega)$$

Substituting this result into the full continuity equation yields the following expression:

$$\boxed{\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho u_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\omega}(\rho u_\omega) = 0}$$

Problem 1.6

From Appendix A, the following value is obtained for the convective derivative:

$$(\mathbf{a}\bullet\nabla)\,\mathbf{a} = \frac{1}{h_1}\left\{\left(a_1\frac{\partial a_1}{\partial x_1} + a_2\frac{\partial a_2}{\partial x_1} + a_3\frac{\partial a_3}{\partial x_1}\right) - \frac{a_2}{h_2}\left[\frac{\partial}{\partial x_1}(h_2\,a_2) - \frac{\partial}{\partial x_2}(h_1\,a_1)\right]\right.$$
$$\left. + \frac{a_3}{h_3}\left[\frac{\partial}{\partial x_3}(h_1\,a_1) - \frac{\partial}{\partial x_1}(h_3\,a_3)\right]\right\}\mathbf{e}_1 + (\bullet\bullet\bullet)\mathbf{e}_2 + (\bullet\bullet\bullet)\mathbf{e}_3$$

Applying this result to cylindrical coordinates, we interpret the various terms as follows:

$$\begin{array}{lll} a_1 = u_R & a_2 = u_\theta & a_3 = u_z \\ x_1 = R & x_2 = \theta & x_3 = z \\ h_1 = 1 & h_2 = R & h_3 = 1 \\ \mathbf{e}_1 = \mathbf{e}_R & \mathbf{e}_2 = \mathbf{e}_\theta & \mathbf{e}_3 = \mathbf{e}_z \end{array}$$

Using these results, the required term becomes:

$$\mathbf{e}_R \bullet \left[(\mathbf{u}\bullet\nabla)\mathbf{u}\right] = \left(u_R \frac{\partial u_R}{\partial R} + u_\theta \frac{\partial u_\theta}{\partial R} + u_z \frac{\partial u_z}{\partial R} \right) - \frac{u_\theta}{R}\left[\frac{\partial}{\partial R}(R u_\theta) - \frac{\partial u_R}{\partial \theta} \right] + u_z \left(\frac{\partial u_R}{\partial z} - \frac{\partial u_z}{\partial R} \right)$$

Simplifying the right side of this equation produces the required result:

$$\boxed{\mathbf{e}_R \bullet \left[(\mathbf{u}\bullet\nabla)\mathbf{u}\right] = u_R \frac{\partial u_R}{\partial R} + \frac{u_\theta}{R}\frac{\partial u_R}{\partial \theta} - \frac{u_\theta^{\,2}}{R} + u_z \frac{\partial u_R}{\partial z}}$$

Problem 1.7

We use the same starting equation from Appendix A as in the previous problem. However, since we are dealing with spherical coordinates here, the various terms are as follows:

$$\begin{array}{lll} a_1 = u_r & a_2 = u_\theta & a_3 = u_\omega \\ x_1 = r & x_2 = \theta & x_3 = \omega \\ h_1 = 1 & h_2 = r & h_3 = r\sin\theta \\ \mathbf{e}_1 = \mathbf{e}_r & \mathbf{e}_2 = \mathbf{e}_\theta & \mathbf{e}_3 = \mathbf{e}_\omega \end{array}$$

Using these results, the required term becomes:

$$\mathbf{e}_r \bullet \left[(\mathbf{u}\bullet\nabla)\mathbf{u}\right] = \left(u_r \frac{\partial u_r}{\partial r} + u_\theta \frac{\partial u_\theta}{\partial r} + u_\omega \frac{\partial u_\omega}{\partial r} \right) - \frac{u_\theta}{r}\left[\frac{\partial}{\partial r}(r\, u_\theta) - \frac{\partial u_r}{\partial \theta} \right] + \frac{u_\omega}{r\sin\theta}\left(\frac{\partial u_r}{\partial \omega} - \frac{\partial}{\partial r}(r\, u_\omega \sin\theta) \right)$$

Simplifying the right side of this equation yields the required result:

$$\boxed{\mathbf{e}_r \bullet \left[(\mathbf{u}\bullet\nabla)\mathbf{u}\right] = u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^{\,2} + u_\omega^{\,2}}{r} + \frac{u_\omega}{r\sin\theta}\frac{\partial u_r}{\partial \omega}}$$

Problem 1.8

For a Newtonian fluid, the shear stress tensor is defined by the following equation:

$$\tau_{ij} = \lambda\,\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Evaluating the various terms in this expression for Cartesian coordinates (x, y, z) and Cartesian velocity components (u, v, w) yields the following results:

$$\boxed{\begin{aligned}
\tau_{xx} &= \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\,\mu\frac{\partial u}{\partial x}\\
\tau_{yy} &= \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\,\mu\frac{\partial v}{\partial y}\\
\tau_{zz} &= \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\,\mu\frac{\partial w}{\partial z}\\
\tau_{xy} &= \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\\
\tau_{xz} &= \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\\
\tau_{yz} &= \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)
\end{aligned}}$$

For a monotonic gas, the Stokes relation requires that $\lambda = -2\,\mu/3$. Then the relations obtained above assume the following special form:

$$\boxed{\begin{aligned}
\tau_{xx} &= \frac{\mu}{3}\left(4\frac{\partial u}{\partial x} - 2\frac{\partial v}{\partial y} - 2\frac{\partial w}{\partial z}\right)\\
\tau_{yy} &= \frac{\mu}{3}\left(4\frac{\partial v}{\partial y} - 2\frac{\partial u}{\partial x} - 2\frac{\partial w}{\partial z}\right)\\
\tau_{zz} &= \frac{\mu}{3}\left(4\frac{\partial w}{\partial z} - 2\frac{\partial u}{\partial x} - 2\frac{\partial v}{\partial y}\right)\\
\tau_{xy} &= \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\\
\tau_{xz} &= \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\\
\tau_{yz} &= \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)
\end{aligned}}$$

Problem 1.9

For a Newtonian fluid, the dissipation function is defined by the following equation:

$$\Phi = \lambda \left(\frac{\partial u_k}{\partial x_k} \right)^2 + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_j}{\partial x_i}$$

Evaluating the various terms in this equation for the Cartesian coordinates (x, y, z) and the Cartesian velocity components (u, v, w), yields the following value for Φ:

$$\boxed{\begin{aligned} \Phi = \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)^2 &+ 2\mu \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right] \\ &+ \mu \left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 \right] \end{aligned}}$$

For a monotonic gas, the Stokes relation requires that $\lambda = -2\mu/3$. Then the general expression for Φ obtained above assumes the following special form:

$$\boxed{\begin{aligned} \Phi = \mu \Bigg\{ -\frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)^2 &+ 2\left(\frac{\partial u}{\partial x} \right)^2 + 2\left(\frac{\partial v}{\partial y} \right)^2 + 2\left(\frac{\partial w}{\partial z} \right)^2 \\ &+ \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 \Bigg\} \end{aligned}}$$

Problem 1.10

For steady flow of an inviscid and incompressible fluid, but one for which the density is not constant, the two-dimensional governing equations are:

$$\boxed{\begin{aligned} \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) &= 0 \\ \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} \\ \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y} \end{aligned}}$$

Dividing the continuity equation by ρ_0 and using the definitions of the new velocity components as given, we get:

$$\frac{\partial}{\partial x}\left(\sqrt{\frac{\rho}{\rho_0}}\,u^*\right)+\frac{\partial}{\partial y}\left(\sqrt{\frac{\rho}{\rho_0}}\,v^*\right)=0$$

$$\therefore\ \sqrt{\frac{\rho}{\rho_0}}\left(\frac{\partial u^*}{\partial x}+\frac{\partial v^*}{\partial x}\right)+u^*\frac{\partial}{\partial x}\left(\sqrt{\frac{\rho}{\rho_0}}\right)+v^*\frac{\partial}{\partial y}\left(\sqrt{\frac{\rho}{\rho_0}}\right)=0$$

The last two terms in the last equation represent the steady-state form of the material derivative of the square root of the density ratio. For an incompressible fluid, this quantity will be zero. Then the continuity equation becomes:

$$\frac{\partial u^*}{\partial x}+\frac{\partial u^*}{\partial x}=0 \tag{1.15}$$

Adding the original form of the continuity equation to each of the components of the momentum equation, and dividing throughout by the constant ρ_0, yields the following form of the momentum equations:

$$\frac{\partial}{\partial x}\left(\frac{\rho}{\rho_0}u^2\right)+\frac{\partial}{\partial y}\left(\frac{\rho}{\rho_0}u\,v\right)=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\rho}{\rho_0}u\,v\right)+\frac{\partial}{\partial y}\left(\frac{\rho}{\rho_0}v^2\right)=-\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

Using the definitions of the new velocity components, these equations become:

$$\frac{\partial}{\partial x}\left(u^{*2}\right)+\frac{\partial}{\partial y}\left(u^*v^*\right)=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\frac{\partial}{\partial x}\left(u^*v^*\right)+\frac{\partial}{\partial y}\left(v^{*2}\right)=-\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

Expanding the terms on the left side of this equation and using Eq. (1.15) reduces the momentum equations to those of an incompressible fluid. The resulting equations are as follows:

$$\boxed{\begin{gathered}\frac{\partial u^*}{\partial x}+\frac{\partial v^*}{\partial y}=0\\ u^*\frac{\partial u^*}{\partial x}+v^*\frac{\partial u^*}{\partial y}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}\\ u^*\frac{\partial v^*}{\partial x}+v^*\frac{\partial v^*}{\partial y}=-\frac{1}{\rho_0}\frac{\partial p}{\partial y}\end{gathered}}$$

2

FLOW KINEMATICS

Problem 2.1

The following graph was drawn using EXCEL.

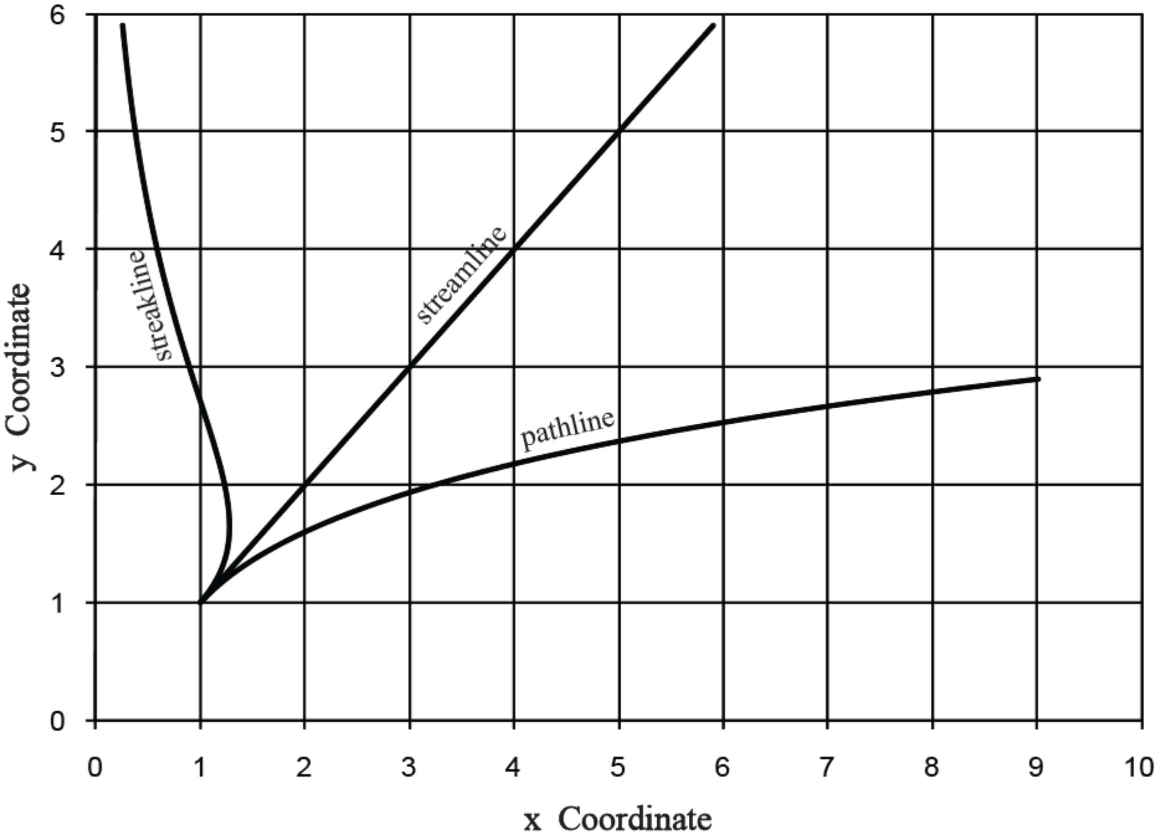


Figure Q.2.1: Streamline, pathline, and streakline for the flow field $u = x(1+2t)$, $v = w = 0$

Problem 2.2

(a)

$$\frac{dy}{dx} = \frac{v}{u} = 1 + t$$

$$\therefore\ y = (1+t)\,x + C$$

$$\text{But} \quad x = y = 1 \quad \text{when} \quad t = 0 \quad \Rightarrow \quad C = 0$$

Hence at $t = 0$ the equation of the streamline is:

$$\boxed{y = x}$$

(b)

$$\frac{dx}{dt} = u = \frac{1}{1+t} \qquad \frac{dy}{dt} = v = 1$$

$$\therefore x = \log(1+t) + C_1 \qquad \therefore y = t + C_2$$

The condition that $x = y = 1$ when $t = 0$ requires that $C_1 = C_2 = 1$, so that:

$$x = 1 + \log(1+t) \qquad y = 1 + t$$

Eliminating t between these two equations shows that the equation of the pathline is:

$$\boxed{y = e^{x-1}}$$

(c) Here, the equations obtained in (b) above are required to satisfy the condition $x = y = 1$ when $t = \tau$. This leads to the values $C_1 = 1 - \log(1+\tau)$ and $C_2 = 1 - \tau$. Hence the parametric equations of the streakline are:

$$x = \log(1+t) + 1 - \log(1+\tau) \qquad y = t + 1 - \tau$$

At time $t = 0$ these equations become:

$$x = 1 - \log(1+\tau) \qquad y = 1 - \tau$$

Eliminating the parameter τ between these two equations yields the following equation for the streakline at $t = 0$:

$$\boxed{y = 2 - e^{1-x}}$$

Problem 2.3

(a)

$$u = x(1+t) \qquad v = 1 \qquad w = 0$$

$$\frac{dx}{ds} = x(1+t) \Rightarrow x = C_1 e^{(1+t)s} = C_1 e^{s} \text{ at } t = 0$$

$$\frac{dy}{ds} = 1 \Rightarrow y = s + C_2$$

But $x = y = 1$ when $s = 0$ so that $C_1 = C_2 = 1$. Hence:

$$x = e^{s} \text{ and } y = s + 1$$

$$\boxed{\therefore\ x = e^{y-1}}$$

(b)
$$\frac{dx}{dt} = x(1+t) \Rightarrow x = C_1 e^{t(1+t/2)}$$
$$\frac{dy}{dt} = 1 \Rightarrow y = t + C_2$$

But $x = y = 1$ when $t = 0$ so that $C_1 = C_2 = 1$. Hence:

$$x = e^{t(1+t/2)} \text{ and } y = t+1$$

$$\boxed{\therefore\ x = e^{(y^2-1)/2}}$$

(c) As in (b): $x = C_1 e^{t(1+t/2)}$ and $y = t + C_2$

But $x = y = 1$ when $t = \tau$ so that $C_1 = e^{-\tau(1+\tau/2)}$ and $C_2 = 1-\tau$. Hence:

$$x = e^{t(1+t/2)-\tau(1+\tau/2)} = e^{-\tau(1+\tau/2)} \text{ for } t = 0$$
$$\text{and} \quad y = 1 + t - \tau = 1 - \tau \text{ for } t = 0$$

$$\boxed{\therefore\ x = e^{-(1-y)(3-y)/2}}$$

(d) From the continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = \frac{D\rho}{Dt} + \rho \nabla \bullet \mathbf{u} = 0$$

Hence: $$\frac{D\rho}{Dt} = -\rho \nabla \bullet \mathbf{u} = -\rho(1+t) \quad \text{since } \nabla \bullet \mathbf{u} = (1+t) \text{ here.}$$

$$\text{i.e. } \frac{D}{Dt}(\log \rho) = -(1+t)$$

$$\text{Therefore} \quad \log \rho = -t(1+t/2) + \log C$$

$$\text{So that} \quad \rho = C e^{-t(1+t/2)}$$

But $\rho = \rho_0$ when $t = 0$ so that $C = \rho_0$. Hence:

$$\boxed{\rho = \rho_0 e^{-t(1+t/2)}} \quad \text{along the streamline}$$

Problem 2.4

The equations that define the streamlines are as follows:

$$\frac{dx_i}{u_i} = ds \qquad or \qquad \frac{dx_i}{x_i} = \frac{ds}{(1+t)}$$

$$\therefore \log x_i = \frac{s}{(1+t)} + \log C_i$$

$$\text{or } x_i = C_i e^{s/(1+t)}$$

Let $x_i = x_{i0}$ when $s = 0$. Then:

$$x_i = x_{i0} \; e^{s/(1+t)}$$

So that: $$\boxed{\frac{x}{x_0} = \frac{y}{y_0} = \frac{z}{z_0}}$$ for any time t.

The equations that define the pathlines are as follows:

$$\frac{dx_i}{dt} = u_i \qquad or \qquad \frac{dx_i}{dt} = \frac{x_i}{(1+t)}$$

$$\therefore \log x_i = \log(1+t) + \log C_i$$

$$\text{or} \;\; x_i = C_i\,(1+t)$$

Let $x_i = x_{i0}$ when $s = 0$. Then:

$$x_i = x_{i0}\,(1+t) \qquad \text{or} \qquad \frac{x_i}{x_{i0}} = (1+t)$$

So that: $$\boxed{\frac{x}{x_0} = \frac{y}{y_0} = \frac{z}{z_0}}$$ as per the streamlines.

Problem 2.5

(a) $$u = 16x^2 + y \qquad v = 10 \qquad w = yz^2$$

On $0 \le x \le 10,\ y = 0$: $$\int_0^{10} u\,dx = \int_0^{10} 16x^2 dx = \frac{16{,}000}{3}$$

On $0 \le y \le 5,\ x = 10$: $$\int_0^{5} v\,dy = \int_0^{5} 10\,dy = 50$$

On $10 \ge x \ge 0,\ y = 5$: $$\int_{10}^{0} u\,dx = \int_{10}^{0} (16x^2 + 5)\,dx = -\frac{16{,}000}{3} - 50$$

On $5 \ge y \le 0,\ x = 0$: $$\int_{5}^{0} v\,dy = \int_{5}^{0} 10\,dy = -50$$

Then, adding the components around the counter-clockwise path gives:

$$\boxed{\Gamma = -50}$$

(b) For the area specified, $\mathbf{n} = \mathbf{e}_z$ and $\omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -1$.

$$\therefore \int_A \boldsymbol{\omega} \bullet \mathbf{n}\,dA = \int_0^{10} dx \int_0^{5} (-1)\,dy$$

$$\boxed{\int_A \boldsymbol{\omega}\bullet\mathbf{n}\,dA = -50}$$

This is the same result that was obtained in (a), so that Eq. (2.5) is verified for this flow.

Problem 2.6

$$u=\frac{x}{x^2+y^2} \qquad \text{and} \qquad v=\frac{y}{x^2+y^2}$$

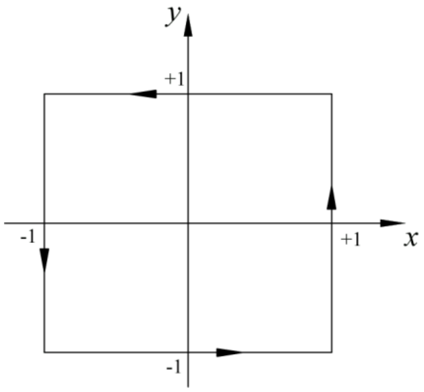
$$\therefore \Gamma = \int_{-1}^{+1} u(x,-1)\,dx + \int_{-1}^{+1} v(+1,y)\,dy + \int_{+1}^{-1} u(x,+1)\,dx + \int_{+1}^{-1} v(-1,y)\,dy$$
$$= \int_{-1}^{+1}\frac{-x\,dx}{x^2+1} + \int_{-1}^{+1}\frac{y\,dy}{1+y^2} + \int_{+1}^{-1}\frac{-x\,dx}{x^2+1} + \int_{+1}^{-1}\frac{y\,dy}{1+y^2}$$

In the foregoing equation, we note that there are two pairs of offsetting integrals. Hence:

$$\boxed{\Gamma = 0}$$

Problem 2.7

(a)



$$u = 9x^2 + 2y \quad v = 10x \quad w = -2yz^2$$

$$\int_{-1}^{+1} u(x,-1)\,dx = \int_{-1}^{+1}(9x^2-2)\,dx = [3x^3-2x]_{-1}^{+1} = 2$$

$$\int_{-1}^{+1} v(+1,y)\,dy = \int_{-1}^{+1} 10\,dy = [10y]_{-1}^{+1} = 20$$

$$\int_{+1}^{-1} u(x,+1)\,dx = \int_{+1}^{-1}(9x^2+2)\,dx = [3x^3+2x]_{+1}^{-1} = -10$$

$$\int_{+1}^{-1} v(-1,y)\,dy = \int_{+1}^{-1}(-10)\,dy = [-10y]_{+1}^{-1} = 20$$

Hence:

$$\boxed{\Gamma = \oint \mathbf{u}\bullet d\mathbf{l} = 2+20-10+20 = 32}$$

(b)

$$\boldsymbol{\omega} = \left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\mathbf{e}_z = -2z^2\,\mathbf{e}_x + 8\,\mathbf{e}_z$$

Hence $\boxed{\boldsymbol{\omega} = -50\,\mathbf{e}_x + 8\,\mathbf{e}_z}$ on the plane $z=5$

(c) For the plane $z = 5$ the unit normal is $\mathbf{n} = \mathbf{e}_z$. Hence:

$$\boxed{\int_A \boldsymbol{\omega} \bullet \mathbf{n}\, dA = \int_{-1}^{+1} dx \int_{-1}^{+1} 8\, dy = 32}$$

This agrees with the result obtained in (a) - as it should, since

$$\Gamma = \oint \mathbf{u} \bullet d\mathbf{l} = \int_A \boldsymbol{\omega} \bullet \mathbf{n}\, dA$$

Problem 2.8

$$u = -\frac{y}{x^2 + y^2} \qquad \text{and} \qquad v = \frac{x}{x^2 + y^2}$$

(a)

$$\begin{aligned}\Gamma &= \int_{-1}^{+1} u(x,-1)\, dx + \int_{-1}^{+1} v(+1,y)\, dy + \int_{+1}^{-1} u(x,+1)\, dx + \int_{+1}^{-1} v(-1,y)\, dy \\ &= \int_{-1}^{+1} \frac{dx}{x^2+1} + \int_{-1}^{+1} \frac{dy}{1+y^2} + \int_{+1}^{-1} \frac{-x\, dx}{x^2+1} + \int_{+1}^{-1} \frac{-dy}{1+y^2} \\ &= 4 \int_{-1}^{+1} \frac{d\alpha}{1+\alpha^2} = 4[\tan^{-1}\alpha]_{-1}^{+1}\end{aligned}$$

$$\boxed{\Gamma = 2\pi}$$

(b)

$$\begin{aligned}\omega &= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \\ &= \left[\frac{1}{(x^2+y^2)} - \frac{2x^2}{(x^2+y^2)^2} \right] + \left[\frac{1}{(x^2+y^2)} - \frac{2y^2}{(x^2+y^2)^2} \right]\end{aligned}$$

$$\boxed{\therefore \omega = 0 \quad \text{provided} \quad x \text{ and } y \neq 0}$$

(c)

$$\frac{\partial u}{\partial x} = \frac{2xy}{(x^2+y^2)^2} \qquad \text{and} \qquad \frac{\partial v}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\boxed{\therefore \nabla \bullet \mathbf{u} = 0 \quad \text{provided} \quad x \text{ and } y \neq 0}$$

Problem 2.9

$$u = \alpha y; \qquad v = \beta x$$

(a)
$$\Gamma = \oint \mathbf{u} \bullet \mathrm{d}\ell = \int_{-1}^{+1} (-\alpha)\, dx + \int_{-1}^{+1} \beta\, dy + \int_{+1}^{-1} \alpha\, dx + \int_{+1}^{-1} (-\beta)\, dy$$
$$= \left[-\alpha x\right]_{-1}^{+1} + \left[\beta y\right]_{-1}^{+1} + \left[\alpha x\right]_{+1}^{-1} + \left[-\beta y\right]_{+1}^{-1}$$
$$= -2\alpha + 2\beta - 2\alpha + 2\beta$$

$$\boxed{\Gamma = -4(\alpha - \beta)}$$

(b)
$$\int_A \omega \bullet n\, dA = \int \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\, dy = \int (\beta - \alpha)\, dx\, dy$$

$$\boxed{\int_A \omega \bullet n\, dA = -4(\alpha - \beta)}$$

(c)
$$\frac{dx}{ds} = \alpha y \quad \text{and} \quad \frac{dy}{ds} = \beta x$$
$$\therefore \frac{dy}{dx} = \frac{\beta x}{\alpha y} \quad \text{or} \quad \alpha y\, dy = \beta x\, dx$$
$$\alpha \frac{y^2}{2} = \beta \frac{x^2}{2} - \frac{c^2}{2}$$

$$\boxed{\beta x^2 - \alpha y^2 = c^2}$$

(d) $\alpha = -1$ and $\beta = +1 \;\Rightarrow\; x^2 + y^2 = c^2$ where $u = -y$ and $v = x$.

But $x = 1$ when $y = 0$ so that $c^2 = 1$. Therefore;

$$\boxed{x^2 + y^2 = 1}$$

(e) $\alpha = \beta = 1 \;\Rightarrow\; x^2 - y^2 = c^2$ where $u = y$ and $v = x$.

But $x = 0$ when $y = 0$ so that $c = 0$. Therefore;

$$\boxed{y = \pm x}$$

Problem 2.10

The vorticity vector will be in the z direction and its magnitude will be:

$$\omega(R,\theta) = \frac{1}{R}\frac{\partial}{\partial R}(R u_\theta) - \frac{1}{R}\frac{\partial u_\theta}{\partial \theta}$$

(a)

$$u_R = 0 \quad \text{and} \quad u_\theta = \omega R$$

$$\frac{\partial}{\partial R}(R u_\theta) = \frac{\partial}{\partial R}(\omega R^2) = 2\omega R$$

$$\text{and} \quad \frac{\partial u_R}{\partial \theta} = 0$$

$$\boxed{\therefore \omega(R,\theta) = 2\omega R}$$

(b)

$$u_R = 0 \quad \text{and} \quad u_\theta = \frac{\Gamma}{2\pi R}$$

$$\frac{\partial}{\partial R}(R u_\theta) = \frac{\partial}{\partial R}\left(\frac{\Gamma}{2\pi}\right) = 0$$

$$\text{and} \quad \frac{\partial u_R}{\partial \theta} = 0$$

$$\boxed{\therefore \omega(R,\theta) = 0 \quad \text{provided} \quad R \neq 0}$$