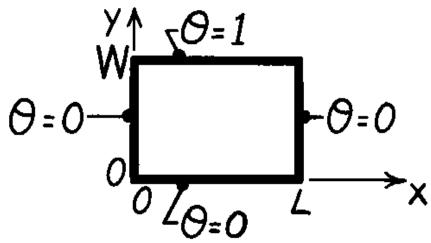
PROBLEM 4.1

KNOWN: Method of separation of variables for two-dimensional, steady-state conduction.

FIND: Show that negative or zero values of λ^2, the separation constant, result in solutions which cannot satisfy the boundary conditions.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties.

ANALYSIS: From Section 4.2, identification of the separation constant λ^2 leads to the two ordinary differential equations, 4.6 and 4.7, having the forms

$$\frac{d^2X}{dx^2}+\lambda^2 X=0 \qquad \frac{d^2Y}{dy^2}-\lambda^2 Y=0 \tag{1,2}$$

and the temperature distribution is $\quad \theta(x,y)=X(x)\cdot Y(y).$ (3)

Consider now the situation when $\lambda^2 = 0$. From Eqs. (1), (2), and (3), find that

$$X=C_1+C_2x, \quad Y=C_3+C_4y \quad \text{and} \quad \theta(x,y)=(C_1+C_2x)\ (C_3+C_4y). \tag{4}$$

Evaluate the constants - C_1, C_2, C_3 and C_4 - by substitution of the boundary conditions:

$$\begin{array}{lll} x=0: & \theta(0,y)=(C_1+C_2\cdot 0)(C_3+C_4\cdot y)=0 & C_1=0 \\ y=0: & \theta(x,0)=(0+C_2\cdot x)(C_3+C_4\cdot 0)=0 & C_3=0 \\ x=L: & \theta(L,0)=(0+C_2\cdot L)(0+C_4\cdot y)=0 & C_2=0 \\ y=W: & \theta(x,W)=(0+0\cdot x)(0+C_4\cdot W)=1 & 0\neq 1 \end{array}$$

The last boundary condition leads to an impossibility ($0 \neq 1$). We therefore conclude that a λ^2 value of zero will not result in a form of the temperature distribution which will satisfy the boundary conditions. Consider now the situation when $\lambda^2 < 0$. The solutions to Eqs. (1) and (2) will be

$$X=C_5e^{-\lambda x}+C_6e^{+\lambda x}, \qquad Y=C_7\cos\lambda y+C_8\sin\lambda y \tag{5,6}$$

and $\quad \theta(x,y)=\left[C_5e^{-\lambda x}+C_6e^{+\lambda x}\right]\ \left[C_7\cos\lambda y+C_8\sin\lambda y\right].$ (7)

Evaluate the constants for the boundary conditions.

$$\begin{array}{lll} y=0: & \theta(x,0)=\left[C_5e^{-\lambda x}+C_6e^{-\lambda x}\right]\ \left[C_7\cos 0+C_8\sin 0\right]=0 & C_7=0 \\ x=0: & \theta(0,y)=\left[C_5e^{0}+C_6e^{0}\right]\ \left[0+C_8\sin\lambda y\right]=0 & C_8=0 \end{array}$$

If $C_8 = 0$, a trivial solution results or $C_5 = -C_6$.

$$x=L: \quad \theta(L,y)=C_5\left[e^{-xL}-e^{+xL}\right]\ C_8\sin\lambda y=0.$$

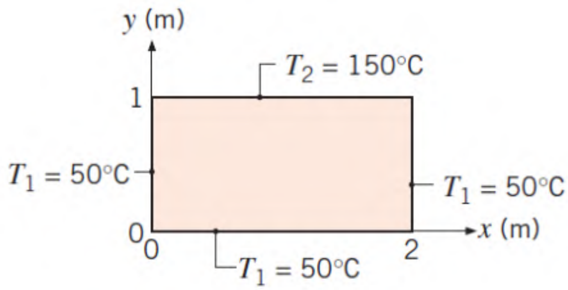
From the last boundary condition, we require C_5 or C_8 is zero; either case leads to a trivial solution with either no x or y dependence.

PROBLEM 4.2

KNOWN: Two-dimensional rectangular plate subjected to prescribed uniform temperature boundary conditions.

FIND: Temperatures along the mid-plane at y = 0.25, 0.5, and 0.75 m considering the first five non-zero terms; assess error resulting from using only first three terms.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties.

ANALYSIS: From Section 4.2, the temperature distribution is

$$\theta(x,y) \equiv \frac{T - T_1}{T_2 - T_1} = \frac{2}{\pi}\sum_{n=1}^{\theta} \frac{(-1)^{n+1}+1}{n} \sin\left(\frac{n\pi x}{L}\right) \cdot \frac{\sinh(n\pi y/L)}{\sinh(n\pi W/L)}. \qquad (1,4.19)$$

Considering now the point (x,y) = (1.0,0.5) and recognizing x/L = 1/2, y/L = 1/4 and W/L = 1/2,

$$\theta(1,0.5) \equiv \frac{T - T_1}{T_2 - T_1} = \frac{2}{\pi}\sum_{n=1}^{\theta} \frac{(-1)^{n+1}+1}{n} \sin\left(\frac{n\pi}{2}\right) \cdot \frac{\sinh(n\pi/4)}{\sinh(n\pi/2)}. \qquad (2)$$

When n is even (2, 4, 6 ...), the corresponding term is zero; hence we need only consider n = 1, 3, 5, 7 and 9 as the first five non-zero terms.

$$\theta(1,0.5) = \frac{2}{\pi}\left\{2\sin\left(\frac{\pi}{2}\right)\frac{\sinh(\pi/4)}{\sinh(\pi/2)} + \frac{2}{3}\sin\left(\frac{3\pi}{2}\right)\frac{\sinh(3\pi/4)}{\sinh(3\pi/2)} + \right.$$

$$\left.\frac{2}{5}\sin\left(\frac{5\pi}{2}\right)\frac{\sinh(5\pi/4)}{\sinh(5\pi/2)} + \frac{2}{7}\sin\left(\frac{7\pi}{2}\right)\frac{\sinh(7\pi/4)}{\sinh(7\pi/2)} + \frac{2}{9}\sin\left(\frac{9\pi}{2}\right)\frac{\sinh(9\pi/4)}{\sinh(9\pi/2)}\right\}$$

$$\theta(1,0.5) = \frac{2}{\pi}[0.755 - 0.063 + 0.008 - 0.001 + 0.000] = 0.445$$

$$T(1,0.5) = \theta(1,0.5)(T_2 - T_1) + T_1 = 0.445(150 - 50) + 50 = 94.5^\circ C.$$ <

Repeating the calculation for y = 0.25 and 0.75 m, the only thing that changes is y/L = 1/8 and 3/8 respectively in the sinh term in the numerator. The results are (repeating the above result for completeness):

$$\theta(1,0.25) = 0.212, \qquad T(1,0.25) = 71.2^\circ C,$$
$$\theta(1,0.50) = 0.445, \qquad T(1,0.25) = 94.5^\circ C,$$ <
$$\theta(1,0.75) = 0.711, \qquad T(1,0.75) = 121^\circ C.$$

Continued...

PROBLEM 4.2 (Cont.)

If only the first three terms of the series, Eq. (2), are considered, the results are:

$$\theta(1, 0.25) = 0.212, \qquad T(1, 0.25) = 71.2^\circ C,$$
$$\theta(1, 0.50) = 0.446, \qquad T(1, 0.25) = 94.6^\circ C, \qquad <$$
$$\theta(1, 0.75) = 0.719, \qquad T(1, 0.75) = 122^\circ C.$$

The worst error is for y = 0.75 m, with only a 0.6% difference.

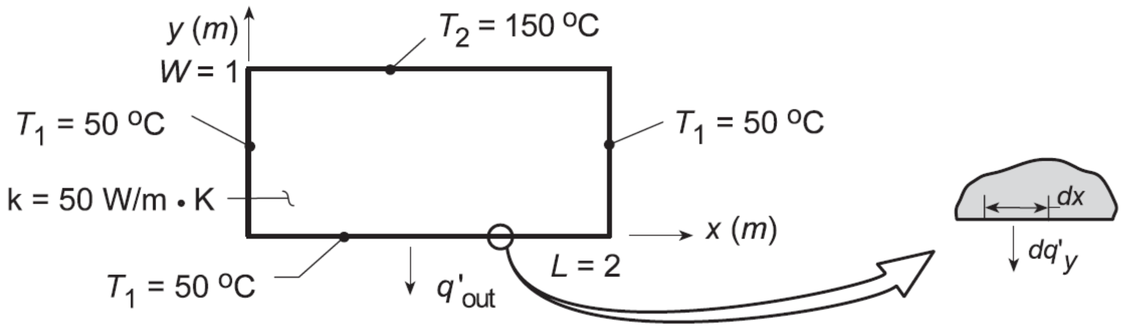
COMMENTS: The number of terms needed for an accurate result depends on the location, with more terms need near those corners where there is a discontinuous change in temperature.

PROBLEM 4.3

KNOWN: Temperature distribution in the two-dimensional rectangular plate of Problem 4.2.

FIND: Expression for the heat rate per unit thickness from the lower surface ($0 \le x \le 2, 0$) and result based on first five non-zero terms of the infinite series.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties.

ANALYSIS: The heat rate per unit thickness *from the plate* along the lower surface is

$$q'_{out} = -\int_{x=0}^{x=2} dq'_y(x,0) = -\int_{x=0}^{x=2} -k\left.\frac{\partial T}{\partial y}\right|_{y=0} dx = k(T_2 - T_1)\int_{x=0}^{x=2}\left.\frac{\partial \theta}{\partial y}\right|_{y=0} dx \qquad (1)$$

where from the solution to Problem 4.2,

$$\theta \equiv \frac{T - T_1}{T_2 - T_1} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\left(\frac{n\pi x}{L}\right)\frac{\sinh(n\pi y/L)}{\sinh(n\pi W/L)}. \qquad (2)$$

Evaluate the gradient of θ from Eq. (2) and substitute into Eq. (1) to obtain

$$q'_{out} = k(T_2 - T_1)\int_{x=0}^{x=2}\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\left(\frac{n\pi x}{L}\right)\left.\frac{(n\pi/L)\cosh(n\pi y/L)}{\sinh(n\pi W/L)}\right|_{y=0} dx$$

$$q'_{out} = k(T_2 - T_1)\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\frac{1}{\sinh(n\pi W/L)}\left[-\cos\left(\frac{n\pi x}{L}\right)\Big|_{x=0}^{2}\right]$$

$$q'_{out} = k(T_2 - T_1)\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\frac{1}{\sinh(n\pi/L)}\left[1 - \cos(n\pi)\right] \qquad <$$

To evaluate the first five, non-zero terms, recognize that since cos(nπ) = 1 for n = 2, 4, 6 ..., only the n-odd terms will be non-zero. Hence,

Continued …

$$q'_{out} = 50\,\text{W/m}\cdot\text{K}\left(150-50\right)^\circ\text{C}\,\frac{2}{\pi}\left\{\frac{(-1)^2+1}{1}\cdot\frac{1}{\sinh(\pi/2)}(2)+\frac{(-1)^4+1}{3}\cdot\frac{1}{\sinh(3\pi/2)}\cdot(2)\right.$$

$$\left.+\frac{(-1)^6+1}{5}\cdot\frac{1}{\sinh(5\pi/2)}(2)+\frac{(-1)^8+1}{7}\cdot\frac{1}{\sinh(7\pi/2)}(2)+\frac{(-1)^{10}+1}{9}\cdot\frac{1}{\sinh(9\pi/2)}(2)\right\}$$

$$q'_{out} = 3.183\,\text{kW/m}\left[1.738+0.024+0.00062+(...)\right]=5.611\,\text{kW/m}$$ <

COMMENTS: If the foregoing procedure were used to evaluate the heat rate into the upper surface, $q'_{in} = -\int_{x=0}^{x=2} dq'_y(x,W)$, it would follow that

$$q'_{in} = k\left(T_2-T_1\right)\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\coth(n\pi/2)\left[1-\cos(n\pi)\right]$$

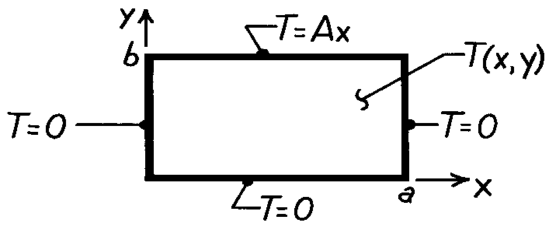
However, with coth(nπ/2) ≥ 1, irrespective of the value of n, and with $\sum_{n=1}^{\infty}\left[(-1)^{n+1}+1\right]/n$ being a divergent series, the complete series does not converge and $q'_{in} \to \infty$. This physically untenable condition results from the temperature discontinuities imposed at the upper left and right corners.

PROBLEM 4.4

KNOWN: Rectangular plate subjected to prescribed boundary conditions.

FIND: Steady-state temperature distribution and heat flux distribution along top edge.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, 2-D conduction, (2) Constant properties.

ANALYSIS: The solution follows the method of Section 4.2. The product solution is

$$T(x,y) = X(x)\cdot Y(y) = \left(C_1\cos\lambda x + C_2\sin\lambda x\right)\left(C_3 e^{-\lambda y} + C_4 e^{+\lambda y}\right)$$

and the boundary conditions are: T(0,y) = 0, T(a,y) = 0, T(x,0) = 0, T(x.b) = Ax. Applying BC#1, T(0,y) = 0, find $C_1 = 0$. Applying BC#2, T(a,y) = 0, find that $\lambda = n\pi/a$ with n = 1,2,…. Applying BC#3, T(x,0) = 0, find that $C_3 = -C_4$. Hence, the product solution is

$$T(x,y) = X(x)\cdot Y(y) = C_2 C_4 \sin\left[\frac{n\pi}{a}x\right]\left(e^{+\lambda y} - e^{-\lambda y}\right).$$

Combining constants and using superposition, find

$$T(x,y) = \sum_{n=1}^{\infty} C_n \sin\left[\frac{n\pi x}{a}\right]\sinh\left[\frac{n\pi y}{a}\right].$$

To evaluate C_n and satisfy BC#4, use orthogonal functions with Equation 4.16 to find

$$C_n = \int_0^a Ax\cdot\sin\left[\frac{n\pi x}{a}\right]\cdot dx/\sinh\left[\frac{n\pi b}{a}\right]\int_0^a \sin^2\left[\frac{n\pi x}{a}\right]dx,$$

noting that y = b. The numerator, denominator and C_n, respectively, are:

$$A\int_0^a x\cdot\sin\frac{n\pi x}{a}\cdot dx = A\left[\left[\frac{a}{n\pi}\right]^2\sin\left[\frac{n\pi x}{a}\right] - \frac{ax}{n\pi}\cos\left[\frac{n\pi x}{a}\right]\right]_0^a = \frac{Aa^2}{n\pi}\left[-\cos(n\pi)\right] = \frac{Aa^2}{n\pi}(-1)^{n+1},$$

$$\sinh\left[\frac{n\pi b}{a}\right]\int_0^a \sin^2\frac{n\pi x}{a}\cdot dx = \sinh\left[\frac{n\pi b}{a}\right]\left[\frac{1}{2}x - \frac{a}{4n\pi}\sin\left[\frac{2n\pi x}{a}\right]\right]_0^a = \frac{a}{2}\cdot\sinh\left[\frac{n\pi b}{a}\right],$$

$$C_n = \frac{Aa^2}{n\pi}(-1)^{n+1}/\frac{a}{2}\sinh\left[\frac{n\pi b}{a}\right] = 2Aa\,(-1)^{n+1}/n\pi\,\sinh\left[\frac{n\pi b}{a}\right].$$

Hence, the temperature distribution is

$$T(x,y) = \frac{2\,Aa}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\sin\left[\frac{n\pi x}{a}\right]\frac{\sinh\left[\frac{n\pi y}{a}\right]}{\sinh\left[\frac{n\pi b}{a}\right]}. \qquad <$$

Continued...

The heat flux normal to the top edge can be found from Fourier's Law:

$$q''_y(x, y=b) = -k\left.\frac{\partial T}{\partial y}\right|_{y=b} = \frac{2Aa}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\sin\left[\frac{n\pi x}{a}\right]\frac{\frac{n\pi}{a}\cosh\left[\frac{n\pi b}{a}\right]}{\sinh\left[\frac{n\pi b}{a}\right]}$$

$$q''_y(x, y=b) = 2A\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\tanh\left[\frac{n\pi b}{a}\right]}\cdot\sin\left[\frac{n\pi x}{a}\right].$$ <

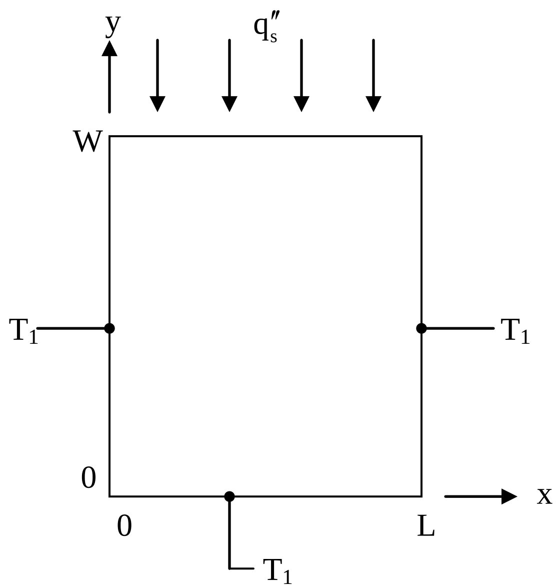
COMMENTS: The heat transfer rate out of the top edge can be found by integrating the heat flux expression above over the x-direction. If this is done similarly for all four surfaces, and the results summed, the net heat transfer rate leaving the rectangle must come out to zero since it is at steady-state with no heat generation.

PROBLEM 4.5

KNOWN: Boundary conditions on four sides of a rectangular plate.

FIND: Temperature distribution.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties.

ANALYSIS: This problem differs from the one solved in Section 4.2 only in the boundary condition at the top surface. Defining $\theta = T - T_\infty$, the differential equation and boundary conditions are

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$$

$$\theta(0,y) = 0 \qquad \theta(L,y) = 0 \qquad \theta(x,0) = 0 \qquad k\left.\frac{\partial\theta}{\partial y}\right|_{y=W} = q_s'' \tag{1a,b,c,d}$$

The solution is identical to that in Section 4.2 through Equation (4.11),

$$\theta = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} \sinh\frac{n\pi y}{L} \tag{2}$$

To determine C_n, we now apply the top surface boundary condition, Equation (1d). Differentiating Equation (2) yields

Continued…

PROBLEM 4.5 (Cont.)

$$\left.\frac{\partial\theta}{\partial y}\right|_{y=W} = \sum_{n=1}^{\infty} C_n \frac{n\pi}{L} \sin\frac{n\pi x}{L} \cosh\frac{n\pi W}{L} \qquad (3)$$

Substituting this into Equation (1d) results in

$$\frac{q_s''}{k} = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \qquad (4)$$

where $A_n = C_n(n\pi/L)\cosh(n\pi W/L)$. The principles expressed in Equations (4.13) through (4.16) still apply, but now with reference to Equation (4) and Equation (4.14), we should choose $f(x) = q_s''/k$, $g_n(x) = \sin\frac{n\pi x}{L}$. Equation (4.16) then becomes

$$A_n = \frac{\frac{q_s''}{k}\int_0^L \sin\frac{n\pi x}{L}dx}{\int_0^L \sin^2\frac{n\pi x}{L}dx} = \frac{q_s''}{k}\frac{2}{\pi}\frac{(-1)^{n+1}+1}{n}$$

Thus

$$C_n = 2\frac{q_s'' L}{k}\frac{(-1)^{n+1}+1}{n^2\pi^2\cosh(n\pi W/L)} \qquad (5)$$

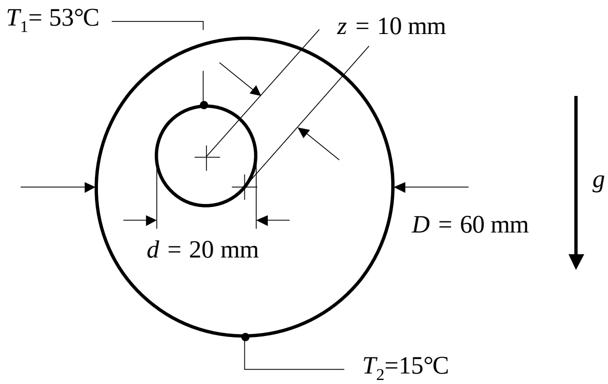
The solution is given by Equation (2) with C_n defined by Equation (5).

PROBLEM 4.6

KNOWN: Diameters and temperatures of horizontal circular cylinders. Eccentricity factor. Heat transfer rate per unit length. Fluid thermal conductivity.

FIND: Effective thermal conductivity.

SCHEMATIC:



ASSUMPTIONS: (1) Constant properties, (2) Steady state conditions.

PROPERTIES: Given: $k = 0.255$ W/m·K.

ANALYSIS: In the absence of free convection the conduction heat transfer per unit length may be found by using the shape factor expression and applying Case 7 of Table 4.1. Hence

$$q'_{\text{cond}} = \frac{S}{L}k\left(T_1 - T_2\right) = \frac{2\pi k\left(T_1 - T_2\right)}{\cosh^{-1}\left(\dfrac{D^2 + d^2 - 4z^2}{2Dd}\right)} = \frac{2\pi \times 0.255 \text{ W/m}\cdot\text{K}\left(53 - 15\right)°\text{C}}{\cosh^{-1}\left(\dfrac{\left(60\times10^{-3}\text{m}\right)^2 + \left(20\times10^{-3}\text{ m}\right)^2 - 4\times\left(10\times10^{-3}\text{ m}\right)^2}{2\times60\times10^{-3}\text{ m}\times20\times10^{-3}\text{ m}}\right)}$$

$$= 63.3 \text{ W/m}$$

The free convection heat transfer rate is

$$q'_{\text{conv}} = \frac{S}{L}k_{\text{eff}}\left(T_1 - T_2\right) = 110 \text{ W/m}$$

Therefore the effective thermal conductivity is

$$k_{\text{eff}} = k\frac{q_{\text{conv}}}{q_{\text{cond}}} = 0.255 \text{ W/m}\cdot\text{K}\cdot\frac{110 \text{ W/m}}{63.3 \text{ W/m}} = 0.44 \text{ W/m}\cdot\text{K}$$ <

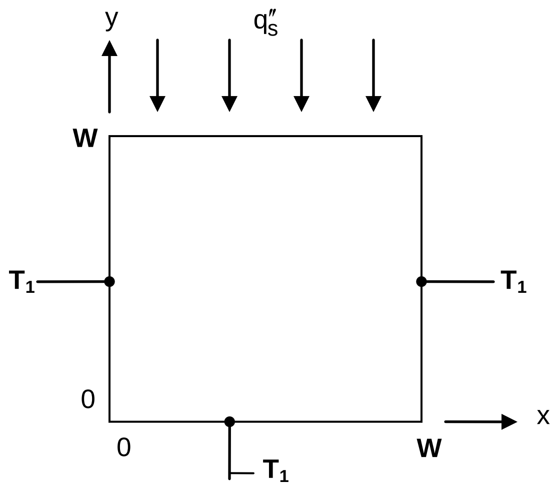
COMMENTS: Buoyancy-induced fluid motion increases the heat transfer rate between the cylinders by 74%.

PROBLEM 4.7

KNOWN: Boundary conditions on four sides of a square plate.

FIND: Expressions for shape factors associated with the *maximum* and *average* top surface temperatures. Values of these shape factors. The maximum and average temperatures for specified conditions.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties.

ANALYSIS: We must first find the temperature distribution as in Problem 4.5. Problem 4.5 differs from the problem solved in Section 4.2 only in the boundary condition at the top surface. Defining $\theta = T - T_\infty$, the differential equation and boundary conditions are

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$$

$$\theta(0,y)=0 \qquad \theta(L,y)=0 \qquad \theta(x,0)=0 \qquad k\left.\frac{\partial\theta}{\partial y}\right|_{y=W} = q_s'' \tag{1a,b,c,d}$$

The solution is identical to that in Section 4.2 through Equation (4.11),

$$\theta = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L} \tag{2}$$

To determine C_n, we now apply the top surface boundary condition, Equation (1d). Differentiating Equation (2) yields

$$\left.\frac{\partial\theta}{\partial y}\right|_{y=W} = \sum_{n=1}^{\infty} C_n \frac{n\pi}{L}\sin\frac{n\pi x}{L}\cosh\frac{n\pi W}{L} \tag{3}$$

Continued …

Substituting this into Equation (1d) results in

$$\frac{q_s''}{k} = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \tag{4}$$

where $A_n = C_n(n\pi/L)\cosh(n\pi W/L)$. The principles expressed in Equations (4.13) through (4.16) still apply, but now with reference to Equation (4) and Equation (4.14), we should choose $f(x) = q_s''/k$, $g_n(x) = \sin\frac{n\pi x}{L}$. Equation (4.16) then becomes

$$A_n = \frac{\frac{q_s''}{k}\int_0^L \sin\frac{n\pi x}{L}dx}{\int_0^L \sin^2\frac{n\pi x}{L}dx} = \frac{q_s''}{k}\frac{2}{\pi}\frac{(-1)^{n+1}+1}{n}$$

Thus

$$C_n = 2\frac{q_s''L}{k}\frac{(-1)^{n+1}+1}{n^2\pi^2\cosh(n\pi W/L)} \tag{5}$$

The solution is given by Equation (2) with C_n defined by Equation (5). We now proceed to evaluate the shape factors.

(a) The maximum top surface temperature occurs at the midpoint of that surface, x = W/2, y = W. From Equation (2) with L = W,

$$\theta(W/2,W) = T_{2,max} - T_1 = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{2}\sinh n\pi = \sum_{n\ odd} C_n (-1)^{(n-1)/2}\sinh n\pi$$

where

$$C_n = 2\frac{q_s''W}{k}\frac{(-1)^{n+1}+1}{n^2\pi^2\cosh n\pi}$$

Thus

$$S_{max} = \frac{q_s''Wd}{k(T_{2,max} - T_1)} = \left[\frac{2}{d}\sum_{n\ odd}\frac{(-1)^{n+1}+1}{n^2\pi^2}(-1)^{(n-1)/2}\tanh n\pi\right]^{-1} = \left[\frac{4}{d}\sum_{n\ odd}\frac{(-1)^{(n-1)/2}}{n^2\pi^2}\tanh n\pi\right]^{-1} \quad <$$

where d is the depth of the rectangle into the page.

Continued…

PROBLEM 4.7 (Cont.)

(b) The average top surface temperature is given by

$$\bar{\theta}(y = W) = \bar{T}_2 - T_1 = \sum_{n=1}^{\infty} C_n \frac{1}{W} \int_0^W \sin \frac{n\pi x}{W}\, dx \sinh n\pi = \sum_{n=1}^{\infty} C_n \frac{1 - (-1)^n}{n\pi} \sinh n\pi$$

Thus

$$\bar{S} = \frac{q_s'' W d}{k(\bar{T}_2 - T_1)} = \left[\frac{2}{d} \sum_{n=1}^{\infty} \frac{[(-1)^{n+1} + 1][1 - (-1)^n]}{n^3 \pi^3} \tanh n\pi \right]^{-1} = \left[\frac{8}{d} \sum_{n\ \mathrm{odd}} \frac{1}{n^3 \pi^3} \tanh n\pi \right]^{-1} \qquad <$$

(c) Evaluating the expressions for the shape factors yields

$$\frac{S_{max}}{d} = \left[4 \sum_{n\ \mathrm{odd}} \frac{(-1)^{(n-1)/2}}{n^2 \pi^2} \tanh n\pi \right]^{-1} = 2.70 \qquad <$$

$$\frac{\bar{S}}{d} = \left[8 \sum_{n\ \mathrm{odd}} \frac{1}{n^3 \pi^3} \tanh n\pi \right]^{-1} = 3.70 \qquad <$$

The temperatures can then be found from

$$T_{2,max} = T_1 + \frac{q}{S_{max} k} = T_1 + \frac{q_s'' W d}{S_{max} k} = 0°C + \frac{1000\ \mathrm{W/m^2} \times 0.01\ \mathrm{m}}{2.70 \times 20\ \mathrm{W/m \cdot K}} = 0.19°C \qquad <$$

$$\bar{T}_2 = T_1 + \frac{q}{\bar{S} k} = T_1 + \frac{q_s'' W d}{\bar{S} k} = 0°C + \frac{1000\ \mathrm{W/m^2} \times 0.01\ \mathrm{m}}{3.70 \times 20\ \mathrm{W/m \cdot K}} = 0.14°C \qquad <$$

PROBLEM 4.8

KNOWN: Heat generation in a buried spherical container.

FIND: (a) Outer surface temperature of the container, (b) Representative isotherms and heat flow lines.

SCHEMATIC:

ASSUMPTIONS: (1) Steady-state conditions, (2) Soil is a homogeneous medium with constant properties.

PROPERTIES: *Table A-3*, Soil (300K): k = 0.52 W/m·K.

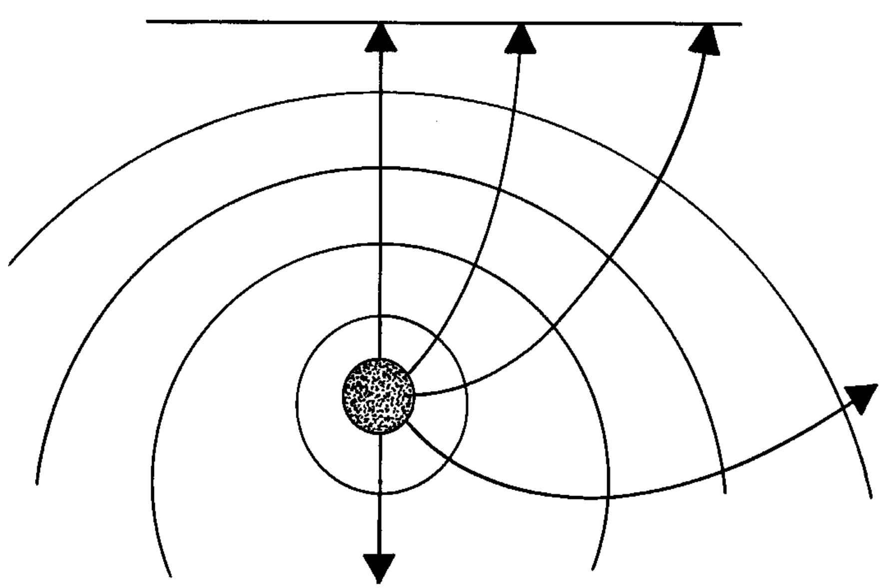
ANALYSIS: (a) From an energy balance on the container, $q = \dot{E}_g$ and from the first entry in Table 4.1,

$$q = \frac{2\pi D}{1 - D/4z} k\left(T_1 - T_2\right).$$

Hence,

$$T_1 = T_2 + \frac{q}{k}\frac{1 - D/4z}{2\pi D} = 20^\circ C + \frac{500W}{0.52\frac{W}{m \cdot K}}\frac{1 - 2m/40m}{2\pi\left(2m\right)} = 92.7^\circ C$$ <

(b) The isotherms may be viewed as spherical surfaces whose center moves downward with increasing radius. The surface of the soil is an isotherm for which the center is at z = ∞.

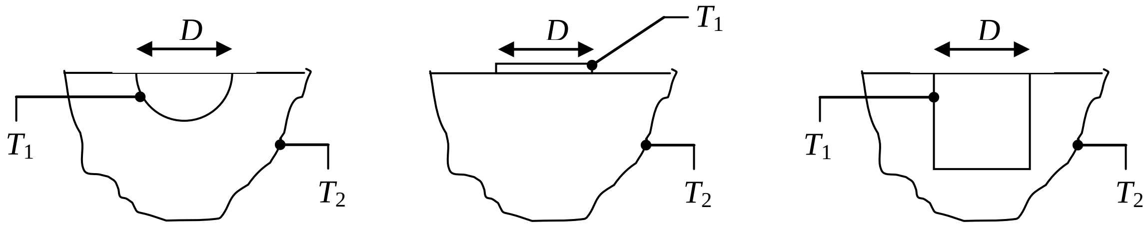


PROBLEM 4.9

KNOWN: Shape of objects at surface of semi-infinite medium.

FIND: Shape factors between object at temperature T_1 and semi-infinite medium at temperature T_2.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, (2) Medium is semi-infinite, (3) Constant properties, (4) Surface of semi-infinite medium is adiabatic.

ANALYSIS: Cases 12 -15 of Table 4.1 all pertain to objects buried in an infinite medium. Since they all possess symmetry about a horizontal plane that bisects the object, they are equivalent to the cases given in this problem for which the horizontal plane is adiabatic. In particular, the heat flux is the same for the cases of this problem as for the cases of Table 4.1. Note, that when we use Table 4.1 to determine the dimensionless conduction heat rate, q_{ss}^*, we must be consistent and use the surface area of the "entire" object of Table 4.1, not the "half" object of this problem. Then

$$q'' = \frac{q}{A_s} = \frac{q_{ss}^* k(T_1 - T_2)}{L_c}$$

where $L_c = (A_s / 4\pi)^{1/2}$ and A_s is the area given in Table 4.1

When we calculate the shape factors we must account for the fact that the surface areas and heat transfer rates for the objects of this problem are half as much as for the objects of Table 4.1.

$$S = \frac{q}{k(T_1 - T_2)} = \frac{q'' A_s/2}{k(T_1 - T_2)} = \frac{q_{ss}^* A_s}{2L_c} = \frac{q_{ss}^* (4\pi A_s)^{1/2}}{2}$$

where A_s is still the area in table 4.1 and the 2 in the denominator accounts for the area being halved. Thus, finally,

$$S = q_{ss}^* (\pi A_s)^{1/2}$$

(a) $\quad S = 1 \cdot (\pi \cdot \pi D^2)^{1/2} = \pi D$ <

(b) $\quad S = \dfrac{2\sqrt{2}}{\pi}\left(\pi \cdot \dfrac{\pi D^2}{2}\right)^{1/2} = 2D$ <

This agrees with Table 4.1a, Case 10.

(c) $\quad S = 0.932(\pi \cdot 2D^2)^{1/2} = \sqrt{2\pi}(0.932)D = 2.34D$ <

(d) The height of the "whole object" is d = 2D. Thus

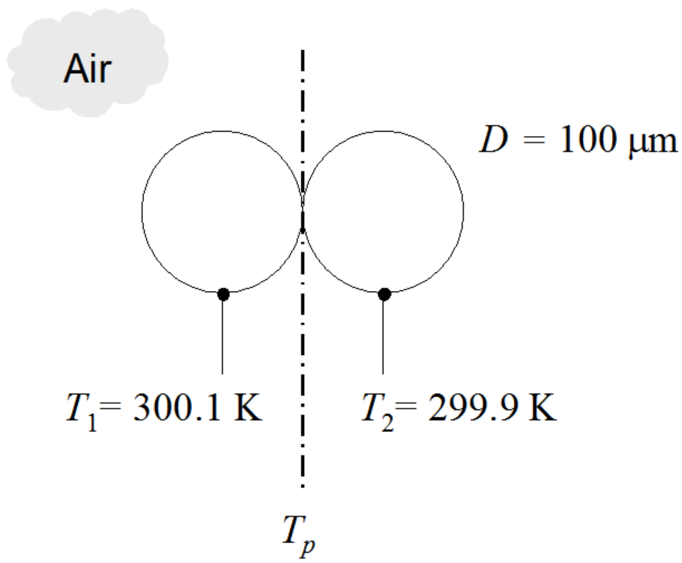
$$S = 0.961\left[\pi\left(2D^2 + 4D \cdot 2D\right)\right]^{1/2} = \sqrt{10\pi}(0.961)D = 5.39D$$ <

PROBLEM 4.10

KNOWN: Diameters and temperatures of spherical particles that are in contact.

FIND: Heat transfer rate.

SCHEMATIC:



ASSUMPTIONS: (1) Constant properties, (2) Isothermal particles.

PROPERTIES: Table A.4, Air (300 K): k = 0.0263 W/m·K.

ANALYSIS: By symmetry, the vertical plane at the particle contact point is at temperature $T_p = (T_1 + T_2)/2 = 300$ K. Therefore, conduction between the particles q_{12} is equal to conduction from particle 1 to the plane, $q_{1p} = Sk(T_1 - T_p)$. The shape factor is that of Case 1 of Table 4.1 evaluated at $z = D/2$ yielding $S = 4\pi D$. Therefore,

$$q_{12} = q_{1p} = 4\pi Dk\left(T_1 - \frac{T_1 + T_2}{2}\right) = 4\pi \times 100 \times 10^{-6}\,\text{m} \times 0.0263\ \text{W/m}\cdot\text{K} \times 0.1\ \text{K} = 3.3 \times 10^{-6}\ \text{W} = 3.3\mu\text{W} \quad <$$

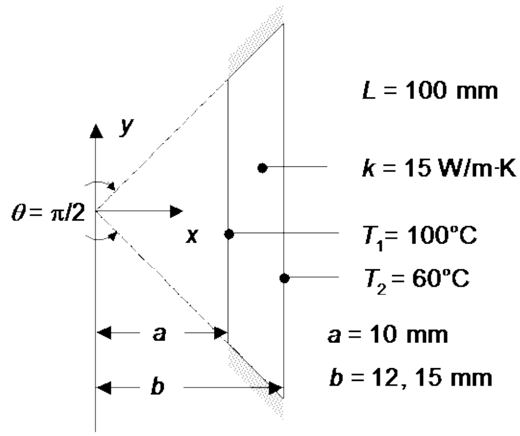
COMMENTS: (1) The air thermal conductivity in the vicinity of the contact point would be reduced by nanoscale effects such as those described in Chapter 2. In applying the shape factor of Case 1 of Table 4.1 to the $z = D/2$ situation we have implicitly assumed that nanoscale effects are negligible. See B. Gebhart, *Heat Conduction and Mass Diffusion*, McGraw-Hill, 1993 for an appropriate treatment of nanoscale phenomena for this geometry. (2) The effective thermal conductivity of porous media composed of high thermal conductivity particles, such as packed metal powder layers, may be estimated by accounting for the particle size and packing distribution and using an analysis such as the one presented here.

PROBLEM 4.11

KNOWN: Dimensions of a two-dimensional object, applied boundary conditions and thermal conductivity.

FIND: (a) Shape factor for the object if the dimensions are a = 10 mm, b = 12 mm. (b) Shape factor for a = 10 mm, b = 15 mm. (c) Shape factor for cases (a) and (b) using the alternative conduction analysis (d) For T_1 = 100°C and T_2 = 60°C, determine the heat transfer rate per unit depth for k = 15 W/m·K for cases (a) and (b).

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties.

PROPERTIES: Given: k = 15 W/m·K.

ANALYSIS: (a) The geometry and applied boundary conditions correspond to Case 11 of Table 4.1(a). Noting that the diagonals of the square channel of Case 11 are adiabats, the shape factor for $b/a = W/w$ = 12/10 = 1.2 is one-fourth the shape factor given in Table 4.1(a),

$$S = 0.25 \times \frac{2\pi L}{0.785 \ln(W/w)} = \frac{2.00L}{\ln(W/w)} = \frac{2.00 \times 0.10\text{m}}{\ln(1.2)} = 1.097 \qquad <$$

(b) For $b/a = W/w$ = 15/10 = 1.5,

$$S = 0.25 \times \frac{2\pi L}{0.930 \ln(W/w) - 0.050} = \frac{1.69L}{\ln(W/w) - 0.0534} = \frac{1.69 \times 0.10\text{m}}{\ln(1.5) - 0.0534} = 0.480 \qquad <$$

(c) From the one-dimensional alternative conduction analysis with the top surface described by $y = x$ and $A = 2yL$,

$$q_x = -kA\frac{dT}{dx} = -2kLx\frac{dT}{dx}$$

Separating variables and integrating yields

$$\int_{x=a}^{b} \frac{dx}{x} = -\frac{2kL}{q_x}\int_{T_1}^{T_2} dT \quad \text{or} \quad \ln(b/a) = \frac{2kL}{q_x}(T_1 - T_2)$$

Hence, $S_{1\text{-D}} = 2L/[\ln(b/a)]$.

Continued…

For $b/a = 1.2$, $S_{1\text{-D}} = 2 \times 0.1\text{m}/[\ln(1.2)] = 1.097$. For $b/a = 1.5$, $S_{1\text{-D}} = 2 \times 0.1\text{m}/[\ln(1.5)] = 0.493$. <

As b/a becomes small, the influence of the lateral edges is diminished and one-dimensional conditions are approached. Hence, the shape factor estimated using the one-dimensional alternative conduction solution is nearly the same as for the two-dimensional shape factor for $b/a = 1.2$. As b/a is increased, the lateral edge effects become more important, and the shape factors obtained by the two methods begin to diverge in value. As two-dimensional conduction in the object becomes more pronounced, the heat transfer rate is decreased relative to that associated with the assumed one-dimensional conditions.

(d) For $b/a = 1.2$, the heat transfer rate is

$$q = Sk(T_1 - T_2) = 1.097 \times 15\ \text{W/m}\cdot\text{K} \times (100 - 60)^\circ\text{C} = 658\ \text{W}$$ <

for $b/a = 1.5$, the heat transfer rate is

$$q = Sk(T_1 - T_2) = 0.480 \times 15\ \text{W/m}\cdot\text{K} \times (100 - 60)^\circ\text{C} = 288\ \text{W}$$ <

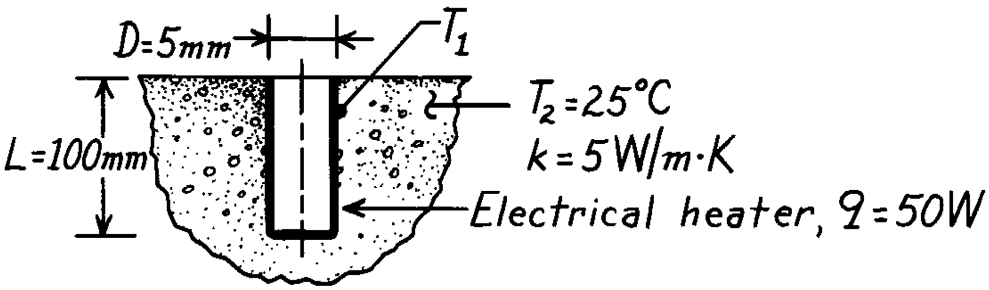
COMMENTS: The heat transfer rate is independent of the individual values of a or b. As either b or a is increased while maintaining a fixed b/a ratio, the cross-sectional area for heat transfer increases, but the increase is offset by increased thickness through which the conduction occurs. The offsetting effects balance one another, and the net result is no change in the heat transfer rate.

PROBLEM 4.12

KNOWN: Electrical heater of cylindrical shape inserted into a hole drilled normal to the surface of a large block of material with prescribed thermal conductivity.

FIND: Temperature reached when heater dissipates 50 W with the block at 25°C.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Block approximates semi-infinite medium with constant properties, (3) Negligible heat loss to surroundings above block surface, (4) Heater can be approximated as isothermal at T_1.

ANALYSIS: The temperature of the heater surface follows from the rate equation written as

$$T_1 = T_2 + q/kS$$

where S can be estimated from the conduction shape factor given in Table 4.1 for a "vertical cylinder in a semi-infinite medium,"

$$S = 2\pi L/\ell n(4L/D).$$

Substituting numerical values, find

$$S = 2\pi \times 0.1\text{m}/\ell n\left[\frac{4 \times 0.1\text{m}}{0.005\text{m}}\right] = 0.143\text{m}.$$

The temperature of the heater is then

$$T_1 = 25°\text{C} + 50\ \text{W}/(5\ \text{W/m·K} \times 0.143\text{m}) = 94.9°\text{C}.$$ <

COMMENTS: (1) Note that the heater has L >> D, which is a requirement of the shape factor expression.

(2) Our calculation presumes there is negligible thermal contact resistance between the heater and the medium. In practice, this would not be the case unless a conducting paste were used.

(3) Since L >> D, assumption (3) is reasonable.

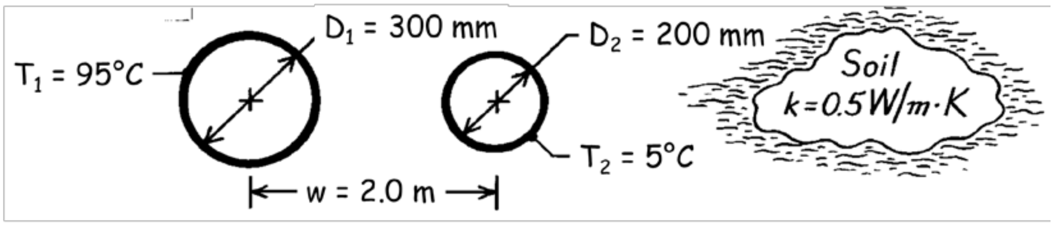
(4) This configuration has been used to determine the thermal conductivity of materials from measurement of q and T_1.

PROBLEM 4.13

KNOWN: Surface temperatures of two parallel pipe lines buried in soil.

FIND: Heat transfer per unit length between the pipe lines.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties, (4) Pipe lines are buried very deeply, approximating burial in an infinite medium, (5) Pipe length >> D_1 or D_2 and w > D_1 or D_2.

ANALYSIS: The heat transfer rate per unit length from the hot pipe to the cool pipe is

$$q' = \frac{q}{L} = \frac{S}{L} k\left(T_1 - T_2\right).$$

The shape factor S for this configuration is given in Table 4.1 as

$$S = \frac{2\pi L}{\cosh^{-1}\left[\dfrac{4w^2 - D_1^2 - D_2^2}{2D_1D_2}\right]}.$$

Substituting numerical values,

$$\frac{S}{L} = 2\pi / \cosh^{-1}\left[\frac{4\times(2.0\text{m})^2 - (0.3\text{m})^2 - (0.2\text{m})^2}{2\times 0.3\text{m}\times 0.2\text{m}}\right] = 2\pi / \cosh^{-1}(132.3)$$

$$\frac{S}{L} = 2\pi / 5.58 = 1.13.$$

Hence, the heat rate per unit length is

$$q' = 1.13\times 0.5\text{W/m}\cdot\text{K}\left(95-5\right)^{\circ}\text{C} = 50.7\ \text{W/m}.$$ <

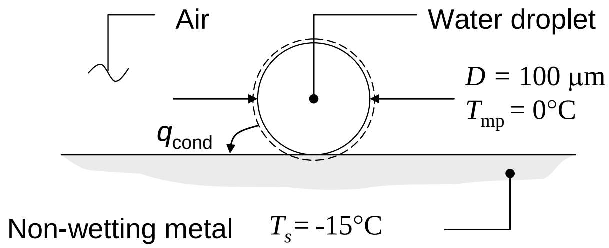
COMMENTS: The heat gain to the cooler pipe line will be larger than 50.7 W/m if the soil temperature is greater than 5°C. How would you estimate the heat gain if the soil were at 25°C?

PROBLEM 4.14

KNOWN: Dimensions and temperature of water droplet.

FIND: Time for droplet to freeze completely.

SCHEMATIC:



ASSUMPTIONS: (1) Constant properties, (2) Negligible convection and radiation, (3) Isothermal water particle, (4) Semi-infinite medium.

PROPERTIES: Table A.4, Air (265 K): k_a = 0.0235 W/m·K. Table A.6, Liquid water (273 K): ρ_w = 1000 kg/m^3.

ANALYSIS: An energy balance on the droplet yields

$$t = \frac{\Delta E}{q_{\text{cond}}} = \frac{V\rho_w h_{sf}}{Sk(T_{\text{mp}} - T_s)} \tag{1}$$

The shape factor S is that of Case 1 of Table 4.1 with $z = D/2$

$$S = \frac{2\pi D}{1 - D/4z} = 4\pi D \tag{2}$$

Combining Equations (1) and (2) with the expression for the droplet volume $V = \pi D^3/6$ yields

$$t = \frac{D^2\rho_w h_{sf}}{24k_a(T_{mp} - T_s)} = \frac{(100\times10^{-6}\text{ m})^2 \times 1000\text{ kg/m}^3 \times 334{,}000\text{ J/kg}}{24\times 0.0235\text{ W/m}\cdot\text{K}\times 15\text{ K}} = 0.39\text{ s}$$ <

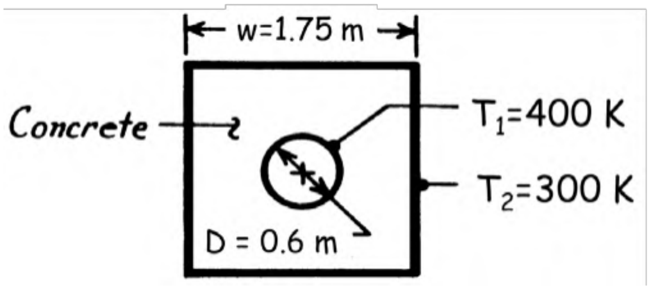
COMMENTS: (1) Solidification might initiate in the lower region of the droplet. The ice that forms would pose an additional conduction resistance between the cold metal surface and the liquid water. This would increase the time needed for the droplet to solidify completely. (2) The air thermal conductivity in the vicinity of the contact point would be reduced by the nanoscale effects described in Chapter 2. In applying this shape factor for the $z = D/2$ case we have implicitly assumed that nanoscale effects are negligible. See B. Gebhart, *Heat Conduction and Mass Diffusion*, McGraw-Hill, 1993.

PROBLEM 4.15

KNOWN: Dimensions and boundary temperatures of a steam pipe embedded in a concrete casing.

FIND: Heat loss per unit length.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Negligible steam side convection resistance, pipe wall resistance and contact resistance (T_1 = 400K), (3) Constant properties.

PROPERTIES: *Table A-3*, Concrete (300K): k = 1.4 W/m·K.

ANALYSIS: The heat rate can be expressed as

$$q = Sk\Delta T_{1\text{-}2} = Sk\left(T_1 - T_2\right)$$

From Table 4.1, the shape factor is

$$S = \frac{2\pi L}{\ell n\left[\frac{1.08\ w}{D}\right]}.$$

Hence,

$$q' = \left[\frac{q}{L}\right] = \frac{2\pi k\left(T_1 - T_2\right)}{\ell n\left[\frac{1.08\ w}{D}\right]}$$

$$q' = \frac{2\pi \times 1.4\text{W/m}\cdot\text{K} \times \left(400 - 300\right)\text{K}}{\ell n\left[\frac{1.08 \times 1.75\text{m}}{0.6\text{m}}\right]} = 767\ \text{W/m}.$$ <

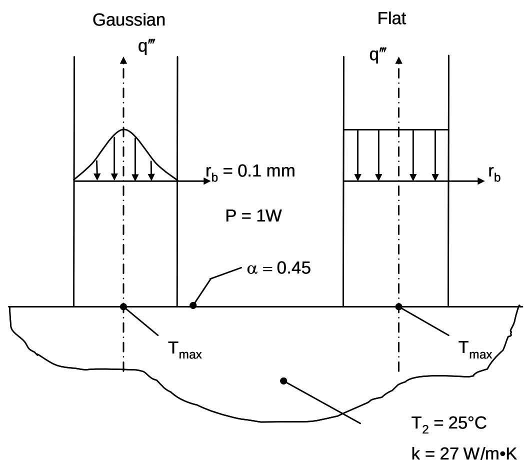
COMMENTS: Having neglected the steam side convection resistance, the pipe wall resistance, and the contact resistance, the foregoing result overestimates the actual heat loss.

PROBLEM 4.16

KNOWN: Power, size and shape of laser beam. Material properties.

FIND: Maximum surface temperature for a Gaussian beam, maximum temperature for a flat beam, and average temperature for a flat beam.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties, (3) Semi-infinite medium, (4) Negligible heat loss from the top surface.

ANALYSIS: The shape factor is defined in Eq. 4.20 and is $q = Sk\Delta T_{1\text{-}2}$ (1)

From the problem statement and Section 4.3, the shape factors for the three cases are:

Beam Shape	Shape Factor	$T_{1,avg}$ or $T_{1,max}$
Gaussian	$2\sqrt{\pi}r_b$	$T_{1,max}$
Flat	πr_b	$T_{1,max}$
Flat	$3\pi^2 r_b/8$	$T_{1,avg}$

For the Gaussian beam, $S_1 = 2\sqrt{\pi} \times 0.1 \times 10^{-3}\text{ m} = 354 \times 10^{-6}\text{ m}$

For the flat beam (max. temperature), $S_2 = \pi \times 0.1 \times 10^{-3}\text{ m} = 314 \times 10^{-6}\text{ m}$

For the flat beam (avg. temperature), $S_3 = (3/8)\times\pi^2 \times 0.1 \times 10^{-3}\text{ m} = 370 \times 10^{-6}\text{ m}$

The temperature at the heated surface for the three cases is evaluated from Eq. (1) as

$$T_1 = T_2 + q/Sk = T_2 + P\alpha/Sk$$

For the Gaussian beam, $T_{1,max} = 25°C + 1\text{ W} \times 0.45 / (354 \times 10^{-6}\text{ m} \times 27\text{ W/m}\cdot\text{K}) = 72.1°C$ <

For the flat beam (T_{max}), $T_{1,max} = 25°C + 1\text{ W} \times 0.45 / (314 \times 10^{-6}\text{ m} \times 27\text{ W/m}\cdot\text{K}) = 78.1°C$ <

For the flat beam (T_{avg}), $T_{1,avg} = 25°C + 1\text{ W} \times 0.45 / (370 \times 10^{-6}\text{ m} \times 27\text{ W/m}\cdot\text{K}) = 70.0°C$ <

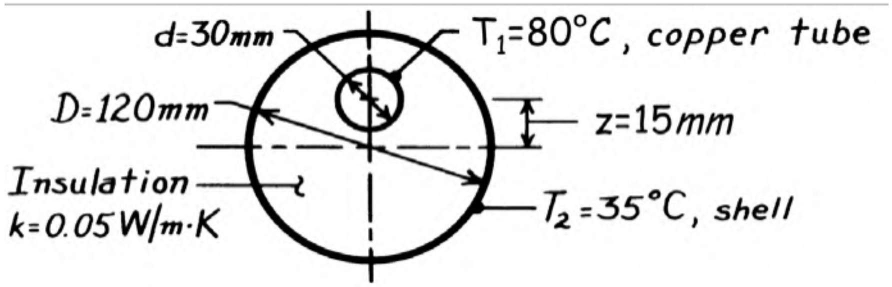
COMMENTS: (1) The maximum temperature occurs at r = 0 for all cases. For the flat beam, the maximum temperature exceeds the average temperature by 78.1 – 70.0 = 8.1 degrees Celsius.

PROBLEM 4.17

KNOWN: Thin-walled copper tube enclosed by an eccentric cylindrical shell; intervening space filled with insulation.

FIND: Heat loss per unit length of tube; compare result with that of a concentric tube-shell arrangement.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties, (3) Thermal resistances of copper tube wall and outer shell wall are negligible, (4) Two-dimensional conduction in insulation.

ANALYSIS: The heat loss per unit length written in terms of the shape factor S is $q' = k(S/\ell)(T_1 - T_2)$ and from Table 4.1 for this geometry,

$$\frac{S}{\ell} = 2\pi / \cosh^{-1}\left[\frac{D^2 + d^2 - 4z^2}{2Dd}\right].$$

Substituting numerical values, all dimensions in mm,

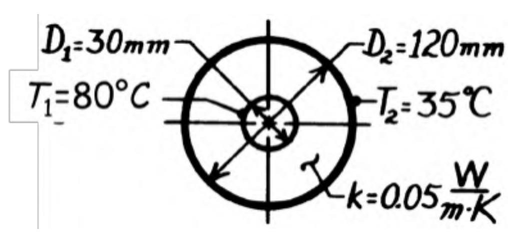
$$\frac{S}{\ell} = 2\pi / \cosh^{-1}\left[\frac{120^2 + 30^2 - 4(15^2)}{2 \times 120 \times 30}\right] = 2\pi / \cosh^{-1}(2.00) = 4.771.$$

Hence, the heat loss is

$$q' = 0.05\text{W/m} \cdot \text{K} \times 4.771(80 - 35)^{\circ}\text{C} = 10.7 \text{ W/m}.$$ <

If the copper tube were concentric with the shell, but all other conditions were the same, the heat loss would be

$$q_c' = \frac{2\pi\ k(T_1 - T_2)}{\ell n(D_2 / D_1)}$$



using Eq. 3.32. Substituting numerical values,

$$q_c' = 2\pi \times 0.05 \frac{\text{W}}{\text{m} \cdot \text{K}} (80 - 35)^{\circ}\text{ C}/\ell n\left({}^{120}/_{30}\right)$$

$$q_c' = 10.2 \text{ W/m}.$$

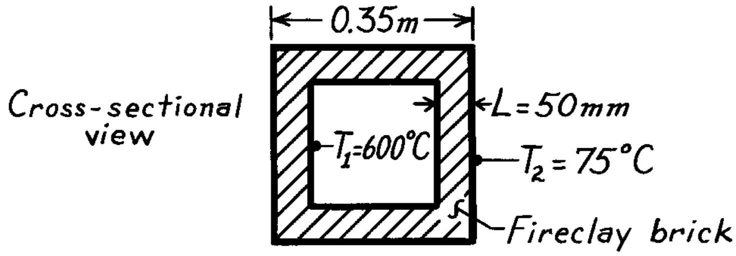
COMMENTS: As expected, the heat loss with the eccentric arrangement is larger than that for the concentric arrangement. The effect of the eccentricity is to increase the heat loss by (10.7–10.2)/10.2 ≈ 5.3%.

PROBLEM 4.18

KNOWN: Cubical furnace, 350 mm external dimensions, with 50 mm thick walls.

FIND: The heat loss, q(W).

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties.

PROPERTIES: *Table A-3*, Fireclay brick $\left(\bar{T}=\left(T_1+T_2\right)/2=610\text{K}\right)$: $k \approx 1.1\ \text{W/m}\cdot\text{K}$.

ANALYSIS: Using relations for the shape factor from Table 4.1,

Plane Walls (6) $S_W=\frac{A}{L}=\frac{0.25\times0.25\text{m}^2}{0.05\text{m}}=1.25\text{m}$

Edges (12) $S_E=0.54D=0.54\times0.25\text{m}=0.14\text{m}$

Corners (8) $S_C=0.15L=0.15\times0.05\text{m}=0.008\text{m}.$

The heat rate in terms of the shape factors is

$$q=kS\left(T_1-T_2\right)=k\left(6S_W+12S_E+8S_C\right)\ \left(T_1-T_2\right)$$
$$q=1.1\frac{\text{W}}{\text{m}\cdot\text{K}}\left(6\times1.25\text{m}+12\times0.14\text{m}+8\times0.008\text{m}\right)\ \left(600-75\right)^\circ\text{C}$$

$$q=5.30\ \text{kW}.$$ <

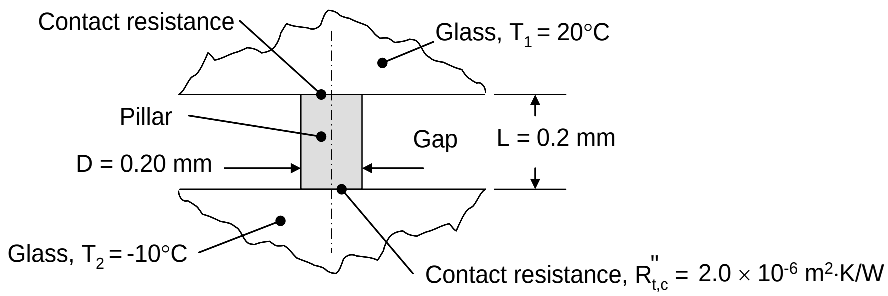
COMMENTS: Note that the restrictions for S_E and S_C have been met.

PROBLEM 4.19

KNOWN: Dimensions of stainless steel pillar and nominal glass temperatures. Contact resistance between pillar and glass.

FIND: Conduction rate through the pillar.

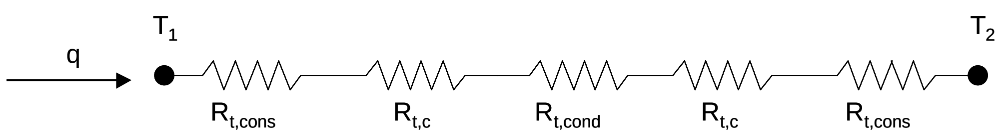
SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties, (3) Negligible radiation, (4) Two-dimensional conduction, (5) Glass behaves as a semi-infinite medium.

PROPERTIES: Table A.1, AISI 302 stainless steel (300 K): k_p = 15.1 W/m·K. Table A.3, plate glass (300 K): k_g = 1.4 W/m·K.

ANALYSIS: Conduction through the pillar results in a depression of the glass temperature adjacent to the pillar. This is associated with a *constriction resistance* within each glass sheet. Therefore, the resistance network consists of two constriction resistances, two contact resistances, and a conduction resistance through the pillar as shown below.



Using the shape factor for Case 10 of Table 4.1(a) the resistances are:

$$R_{t,\text{cons}} = 1/(Sk_g) = 1/(2Dk_g) = 1/(2 \times 0.20 \times 10^{-3}\,\text{m} \times 1.4\ \text{W/m}\cdot\text{K}) = 1786\ \text{K/W}$$

$$R_{t,\text{c}} = R''_{t,c} / A_p = 2.0 \times 10^{-6}\,\text{m}^2 \cdot \text{K/W} / \left[\pi \times \left(0.20 \times 10^{-3}\,\text{m}\right)^2 / 4\right] = 63.66\ \text{K/W}$$

$$R_{t,\text{cond}} = L / k_p A_p = L / k_p \left(\pi D_p^2 / 4\right) = 0.2 \times 10^{-3}\ \text{m} / \left[15.1\ \text{W/m}\cdot\text{K} \times \pi \times \left(0.20 \times 10^{-3}\ \text{m}\right)^2 / 4\right] = 421.6\ \text{K/W}$$

Therefore, the total resistance is

$R_{tot} = 2(R_{t,cons} + R_{t,c}) + R_{t,cond} = 2 \times (1786\ \text{K/W} + 63.66\ \text{K/W}) + 421.6\ \text{K/W} = 4120\ \text{K/W}$

and the conduction through an individual pillar is

$q = (T_1 - T_2)/R_{tot} = [20 - (-10)°\text{C}]/[4120\ \text{K/W}] = 7.28 \times 10^{-3}\ \text{W} = 7.28\ \text{mW}$ <

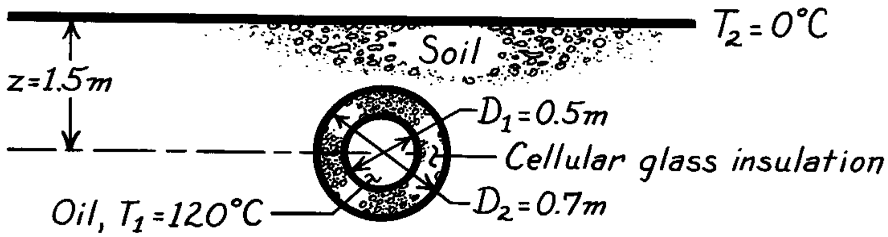
COMMENTS: (1) Constriction of the heat flow within the glass poses the largest resistance to heat transfer. (2) Radiation between the two glass sheets exists, and may be important in determining the overall heat transfer through the window. (3) Extremely high vacuum between the two glass sheets is required to eliminate conduction within the gap. (4) See Manz, Brunner and Wullschleger, "Triple Vacuum Glazing: Heat Transfer and Basic Design Constraints," *Solar Energy*, Vol. 80, pp. 1632-1642, 2006 for more information.

PROBLEM 4.20

KNOWN: Temperature, diameter and burial depth of an insulated pipe.

FIND: Heat loss per unit length of pipe.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction through insulation, two-dimensional through soil, (3) Constant properties, (4) Negligible oil convection and pipe wall conduction resistances.

PROPERTIES: *Table A-3*, Soil (300K): k = 0.52 W/m·K; *Table A-3*, Cellular glass (365K): k = 0.069 W/m·K.

ANALYSIS: The heat rate can be expressed as

$$q = \frac{T_1 - T_2}{R_{tot}}$$

where the thermal resistance is $R_{tot} = R_{ins} + R_{soil}$. From Equation 3.33,

$$R_{ins} = \frac{\ell n\left(D_2 / D_1\right)}{2\pi\ Lk_{ins}} = \frac{\ell n\left(0.7\text{m}/0.5\text{m}\right)}{2\pi\ L \times 0.069\ \text{W/m}\cdot\text{K}} = \frac{0.776\text{m}\cdot\text{K/W}}{L}.$$

From Equation 4.21 and Table 4.1,

$$R_{soil} = \frac{1}{Sk_{soil}} = \frac{\cosh^{-1}\left(2z/D_2\right)}{2\pi\ Lk_{soil}} = \frac{\cosh^{-1}\left(3/0.7\right)}{2\pi \times \left(0.52\ \text{W/m}\cdot\text{K}\right)L} = \frac{0.653}{L}\text{m}\cdot\text{K/W}.$$

Hence,

$$q = \frac{\left(120-0\right)^{\circ}\text{C}}{\frac{1}{L}\left(0.776+0.653\right)\frac{\text{m}\cdot\text{K}}{\text{W}}} = 84\frac{\text{W}}{\text{m}} \times L$$

$$q' = q/L = 84\ \text{W/m}.$$ <

COMMENTS: (1) Contributions of the soil and insulation to the total resistance are approximately the same. The heat loss may be reduced by burying the pipe deeper or adding more insulation.

(2) The convection resistance associated with the oil flow through the pipe may be significant, in which case the foregoing result would overestimate the heat loss. A calculation of this resistance may be based on results presented in Chapter 8.

(3) Since z > 3D/2, the shape factor for the soil can also be evaluated from $S = 2\pi L/\ell n\,(4z/D)$ of Table 4.1, and an equivalent result is obtained.